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On the worldsheet theory of the type IIA AdS$_4$ × $\mathbb{CP}_3$ superstring

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ABSTRACT: We perform a detailed study of the type IIA superstring in AdS$_4 \times \mathbb{CP}_3$. After introducing suitable bosonic light-cone and fermionic kappa worldsheet gauges we derive the pure boson and fermion SU(2|2)×U(1) covariant light-cone Hamiltonian up to quartic order in fields.

As a first application of our derivation we calculate energy shifts for string configurations in a closed fermionic subsector and successfully match these with a set of light-cone Bethe equations.

We then turn to investigate the mismatch between the degrees of freedom of scattering states and oscillatory string modes. Since only light string modes appear as fundamental Bethe roots in the scattering theory, the physical role of the remaining $4_F + 4_B$ massive oscillators is rather unclear. By continuing a line of research initiated by Zarembo, we shed light on this question by calculating quantum corrections for the propagators of the bosonic massive fields. We show that, once loop corrections are incorporated, the massive coordinates dissolve in a continuum state of two light particles.

Contents

1. Introduction

Recently strings on $AdS_4 \times \mathbb{CP}_3$ have enjoyed an increased interest due to the AdS_4 / CFT_3 duality proposed in [1], [2]. The conjecture, nowadays dubbed ABJM duality in the literature, states that a three dimensional $\mathcal{N} = 6$ and SU(N) Chern Simons

theory living on the boundary of AdS_4 are in certain limits dual to type IIA string theory on $AdS_4 \times \mathbb{CP}_3$.

The duality exhibits many shared features with the well studied AdS_5 / CFT_4 correspondence, where perhaps the most striking similarity is the emergence of integrable structures [4], [8], [43]. On the gauge theory side, integrability was demonstrated for the two loop Hamiltonian[1] in [3]. Quickly after, the algebraic curve encoding all the classical solutions at strong and weak coupling together with the all loop asymptotic Bethe equations were put forward in [14], [13], [15]. There after, and under the assumptions of a $SU(2|2)\times U(1)$ symmetry, the exact S matrix were proposed in [42]. Following these findings, a host of various checks and higher order calculations have been performed [16], [17], [18], [19], [20], [21], [22], [28], [32], [23], [24], [25], [30], [31], [43], [29], [33], [34].

That all this has been achieved with such a rapid progress is remarkable since in both dualities the full dynamics can be constructed from symmetry arguments alone. For ABJM, the symmetry group is $OSP(2,2|6)$, which differs quite much from the well known $PSU(2,2|4)$ of AdS_5 / CFT_4. Nevertheless, planar integrability, all loop asymptotic Bethe equations, $SU(2|2)$ scattering and central extension occur in similar ways in both dualities.

In this paper we will perform a detailed study of the string theory side of the ABJM correspondence. Starting from the symmetry group we derive the Lagrangian in a super matrix notation utilizing an uniform light-cone gauge.

As has been demonstrated by Bykov in [27], the symmetry of the gauge fixed string reduces from $OSP(2,2|6)$ to a centrally extended $SU(2|2)\times U(1)$. This is rather similar to the superstring in $AdS_5\times S^5$ which after gauge fixing have a centrally extended $SU(2|2)^2$ algebra [39]. Even though the gauge fixed subalgebras are rather similar, we find that the general structure of the type IIA superstring is considerably more involved than its $AdS_5\times S^5$ cousin.

In order to extract any information from the Lagrangian we need to consider some sort of perturbative expansion. We will make use of a strong coupling expansion, or equivalently, an expansion in number of fields. Utilizing this expansion we derive the pure boson and fermion part of the light-cone Hamiltonian up to quartic order in number of fields [5], [7], [6], [48].

To avoid the rather severe complications of gauge fixing the worldsheet metric, we work in a first order formalism. This has the upshot that the metric components only enters as Lagrange multipliers. However, the theory exhibits higher order fermionic worldsheet time derivatives and to preserve a canonical Poisson structure we need to shift the fermions in a appropriate way. Unfortunately, due to the presence of cubic kinetic terms, this shift adds a 'self interacting' term which is very hard to remove. Not only is the structure complicated, but it also introduces corrections to

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[1]The one loop piece vanishes trivially.

the bosonic momentas. The way we approach this problem is to only present the canonical Hamiltonian for pure boson / fermion fields. We do however present the full light-cone Hamiltonian, prior to the fermionic shift, in the appendix.

Having established the relevant parts of the first order theory, and under the assumption of normal ordering, we calculate energy corrections to a certain set of fermionic string states. Even though the general structure of relevant parts of the Hamiltonian is rather involved, we find that the energy shifts takes a remarkably simple form. This feature was also observed for the bosonic subsector calculated in [26] and seems to be a general feature of the uniform light-cone gauge we imposed. We then match the energy shifts with the predictions coming from a conjectured set of Bethe equations proposed in [13], and rewritten in a light-cone language in [52] and [26]. This is the first calculation that explicitly probes the higher order fermionic sectors, and thus test the two body factorization implied by integrability, of the AdS$_4$ / CFT$_3$ duality[2].

After this we turn to investigate the role of the massive modes of the theory. At the quadratic level the string oscillators come in $4_F + 4_B$ heavy and light modes respectively. From the point of view of the conjectured exact scattering theory [38], the fundamental excitations in the S matrix are the light modes, leaving us with a miss match between the degrees of freedom.

In [11] Zarembo calculated the loop corrections for a massive bosonic mode. There it was found that when quantum corrections are taken into account, the analytic properties of the propagator changes. What happens is that the pole gets shifted onto the branch cut and vanishes. Therefore the heavy mode is not fundamental but rather a composite continuum state of two light particles.

We continue this line of research by showing that exactly the same thing happens with the remaining massive bosons. Even though we do not calculate it explicitly, we also provide some general arguments for why the same thing should happen with the remaining massive fermionic coordinates.

The paper is organized as follows; We start out in section two by presenting some general facts about the (super)matrix representation of the $\mathfrak{osp}(2,2|6)$ algebra. Then by making use of the $\mathbb{Z}_4$ grading of the algebra, we construct the exact string Lagrangian in a convenient kappa and light-cone gauge. In section three we expand the derived theory in a strong coupling limit, equivalent to a near plane-wave expansion, to quartic order. We find that the theory exhibits higher order time derivatives of the fermions, and thus naively introduces a complicated Poisson structure. To tackle this problem, we follow [48] and introduce a fermionic shift with the property that it removes the higher order kinetic terms. Sadly, this shift comes with the price of adding additional cubic and quartic terms to the interacting Hamiltonian. In section four we turn to a perturbative analysis of the string spectrum by calculating energy

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[2]Where we with higher order mean operators constituted of an arbitrary number of fermionic excitations.

shifts for fermionic states. These we then match with a set of uniform light-cone Bethe equations, finding perfect agreement. The last analysis we perform is to calculate loop diagrams for the bosonic heavy modes in section six. We show that all the massive bosonic modes dissolve into a two particle continuum, and therefore, do not appear as fundamental excitations of the scattering theory.

We end the paper with a short summary and outlook together with several appendices where notation and various computational details are presented.

2. Type IIA superstring on $\mathbf{AdS_4 \times \mathbb{CP}_3}$

One of the most beautiful and effective ways to describe a physical theory is through the use of its symmetries. An especially nice approach using algebraic properties of a certain type of string configurations has been developed by Arutyunov and Frolov, see [36] for a nice review. In the below we will apply this procedure for a supersymmetric $AdS_4 \times \mathbb{CP}_3$ string propagating on the supergroup manifold [9] [10]

$$\frac{OSP(2,2|6)}{SO(1,3) \times U(3)}. \tag{2.1}$$

A crucial ingredient is the existence of a $\mathbb{Z}_4$ grading of the symmetry algebra which allows for a construction of the string Lagrangian directly from its graded components [37].

To illustrate the procedure, we begin this section by reviewing some basic facts of the super algebra $\mathfrak{osp}(2,2|6)$.

2.1 Matrix realization of $\mathfrak{osp}(2,2|6)$

The super Lie algebra $\mathfrak{osp}(2,2|6)$ can be represented by 10×10 matrices of the form

$$M = \begin{pmatrix} X_{4\times4} & \theta_{4\times6} \\ \eta_{6\times4} & Y_{6\times6} \end{pmatrix}$$

where X and Y are even matrices whereas θ and η are Grassmannian odd.

To single out the algebra of interest, the matrices M has to satisfy the following reality and transposition rules,

$$M^{st} \begin{pmatrix} C_4 & 0 \\ 0 & \mathbb{1}_{6\times6} \end{pmatrix} + \begin{pmatrix} C_4 & 0 \\ 0 & \mathbb{1}_{6\times6} \end{pmatrix} M = 0$$

$$M^{\dagger} \begin{pmatrix} \Gamma^0 & 0 \\ 0 & -\mathbb{1}_{6\times6} \end{pmatrix} + \begin{pmatrix} \Gamma^0 & 0 \\ 0 & -\mathbb{1}_{6\times6} \end{pmatrix} M = 0$$

where the charge conjugation matrix satisfies $C_4^2 = -\mathbb{1}_{4\times4}$ and Γ_0 is one of the AdS_4 Γ-matrices. In the first appendix we collect all the various matrices encountered in

this section. The super transpose is defined as

$$M^{st} = \begin{pmatrix} X^t & -\eta^t \\ \theta^t & Y^t \end{pmatrix},$$

and the above reality and transposition rules imply

$$X^t = -C_4\,X\,C_4^{-1} \qquad Y^t = -Y, \qquad \eta = -\theta^t\,C_4, \qquad \theta^* = \Gamma^0\,C_4\,\theta. \qquad (2.2)$$

The even X and Y block correspond to the bosonic isometry groups USP(2,2) and SO(6) of AdS_4 and $\mathbb{CP}_3$ respectively. The odd blocks are related by conjugation and constitute 24 real spinor variables. The reality condition on the fermionic block θ relates [3]

$$\theta_{4,i} = \bar{\theta}_{1,i}, \qquad \theta_{3,i} = -\bar{\theta}_{2,i}. \qquad (2.3)$$

As advocated, the super algebra $\mathfrak{osp}(2,2|6)$ admits a $\mathbb{Z}_4$ decomposition as

$$M = M^{(0)} \oplus M^{(2)} \oplus M^{(1)} \oplus M^{(3)}. \qquad (2.4)$$

We want to construct an inner automorphism such that its stationary point coincides with $\mathfrak{so}(1,3) \oplus \mathfrak{u}(3)$. This can be done by introducing two matrices K_4 and K_6

$$K_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad K_6 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

which satisfy $K_4^2 = -\mathbb{1}$ and $K_6^2 = -\mathbb{1}$. These two matrices together with the charge conjugation matrix allows us to define an automorphism as [9]

$$\Omega(M) = \begin{pmatrix} K_4 C_4 & 0 \\ 0 & -K_6 \end{pmatrix} M \begin{pmatrix} K_4 C_4 & 0 \\ 0 & -K_6 \end{pmatrix}^{-1} = \Upsilon\,M\,\Upsilon^{-1},$$

which can be used to construct the different $\mathbb{Z}_4$ components

$$M^{(k)} = \frac{1}{4}\Big(M + i^{3k}\Omega(M) + i^{2k}\Omega^2(M) + i^k\Omega^3(M)\Big), \qquad (2.5)$$

where each component $M^{(k)}$ is an eigenstate of Ω,

$$\Omega(M^{(k)}) = i^k M^{(k)}. \qquad (2.6)$$

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[3]Through out the paper we will denote conjugated objects with bar.

The stationary subalgebra, $M^{(0)}$, coincides with $\mathfrak{so}(1,3) \oplus \mathfrak{u}(3)$ which is the part of $\mathfrak{osp}(2,2|6)$ we want to divide out.

The orthogonal complement $M^{(2)}$ is spanned by matrices satisfying $\Upsilon \, M \, \Upsilon^{-1} = -M$, which boils down to the conditions

$$\{X, \Gamma^5\} = 0, \qquad \{Y, K_6\} = 0. \tag{2.7}$$

These two equations can be solved by

$$X = x_\mu \Gamma^\mu, \qquad Y = Y_i T_i, \tag{2.8}$$

where the first parameterize SO(3,2)/SO(1,3) and the second SO(6)/U(3). For the exact form of the Γ_μ and T_i generators, please consult the appendix.

With this we have established a good parameterization of $\mathfrak{osp}(2,2|6)$. In the next section we will construct the full string Lagrangian from this.

2.2 Group parameterization and string Lagrangian

There are many ways to parameterize OSP(2,2|6) and they are all related through non linear field transformations. In this paper we will use a particulary suitable representation that allows us to fix the bosonic and fermionic worldsheet symmetries in a convenient way [48].

As starting point we introduce the following group element of OSP(2,2|6),

$$G = \Lambda(x^+, x^-) \, f(\eta) \, G_t, \tag{2.9}$$

where the different components are given by

$$\Lambda(x^+, x^-) = \exp\frac{i}{2}(x^+\Sigma_+ + x^-\Sigma_-), \qquad G_t = G_y \, G_{AdS} \, G_{CP}, \qquad f(\eta) = \eta + \sqrt{1+\eta^2}.$$

The $x^\pm = \phi \pm t$ are a light-cone pair constituted of the time and angle coordinate of AdS$_4$ and $\mathbb{CP}_3$ and $\Sigma_\pm$ is the corresponding basis element, $\Sigma_\pm = \pm\Gamma_0 \oplus -i\,T_6$. The fermionic matrix η, entering in $f(\eta)$, is in principle just the odd part of M. The transverse bosonic degrees of freedom are described by G_t,

$$G_t = \begin{pmatrix} G_{AdS} & 0 \\ 0 & G_y \, G_{CP} \end{pmatrix}.$$

The AdS$_4$ part is parameterized by three transverse coordinates, z_i

$$G_{Ads} = \frac{\mathbb{1} + \frac{i}{2}z_i\Gamma^i}{\sqrt{1 - \frac{z_i^2}{4}}}. \tag{2.10}$$

and the G_y element is described by a single real coordinate, y, of the $\mathbb{CP}_3$,

$$G_y = e^{y\,T_5}, \tag{2.11}$$

which is a function of $\cos(y)$ and $\sin(y)$. For the upcoming perturbative analysis it is convenient to relabel the trigonometric functions as

$$\sin(y) \to \frac{1}{2}y, \qquad \cos(y) \to \sqrt{1 - \frac{1}{4}y^2}.$$

The last component of G_t is parameterized by two complex coordinates ω_i (and its conjugate $\bar{\omega}_i$)

$$G_{CP} = \tag{2.12}$$

$$\mathbb{1} + \frac{1}{\sqrt{1 + \frac{1}{4}|w|^2}}(W + \bar{W}) + 4\,\frac{\sqrt{1 + \frac{1}{4}|w|^2} - 1}{|w|^2\sqrt{1 + \frac{1}{4}|w|^2}}(W \cdot \bar{W} + \bar{W} \cdot W),$$

where $W = \frac{1}{2}\omega_i\,\tau_i$ and $|w|^2 = \omega_i\,\bar{\omega}_i$.

Using these parameterizations, we can construct a flat current in $\mathfrak{osp}(2,2|6)$ as

$$\mathcal{A} = \mathcal{A}^{(0)} \oplus \mathcal{A}^{(2)} \oplus \mathcal{A}^{(1)} \oplus \mathcal{A}^{(3)} = -G^{-1}\,dG, \tag{2.13}$$

which components are used to construct the string Lagrangian,

$$\mathscr{L} = -\frac{g}{2}\int d\sigma\,Str\Big(\gamma^{\alpha\beta}\mathcal{A}_\alpha^{(2)}\,\mathcal{A}_\beta^{(2)} + \kappa\,\epsilon^{\alpha\beta}\mathcal{A}_\alpha^{(1)}\,\mathcal{A}_\beta^{(3)}\Big). \tag{2.14}$$

Throughout the paper we will use greek letters for worldsheet indices. The string length parameter is denoted σ and takes values $\sigma \in [-L, L]$. The variable κ in front of the WZ term is demanded by supersymmetry to satisfy[4] $\kappa^2 = 1$. The $\gamma^{\alpha\beta}$ tensor is the Weyl invariant combination of the worldsheet metric with determinant $det\,\gamma^{\alpha\beta} = -1$. Finally, the model is characterized by the string coupling, $g \sim \frac{R^2}{\alpha'}$ with R the radius of the AdS space. This is the only free parameter of the theory and later we will expand the theory in a limit with g taken as large.

Since our aim is to perform a perturbative expansion of the above Lagrangian the gauge fixing procedure gets considerably simplified if we introduce a auxiliary field π which allows us to rewrite the Lagrangian (2.14) as [48]

$$\mathscr{L} = \tag{2.15}$$

$$-g\int d\sigma\,Str\Big(\pi\,\mathcal{A}_0 + \frac{\kappa}{2}\,\epsilon^{\alpha\beta}\,\mathcal{A}_\alpha^{(1)}\,\mathcal{A}_\beta^{(3)} - \frac{1}{2\gamma^{00}}\big(\pi^2 + (\mathcal{A}_1^{(2)})^2\big) + \frac{\gamma^{01}}{\gamma^{00}}\,\pi\,\mathcal{A}_1^{(2)}\Big).$$

Using the equations of motion for π one can easily show that this Lagrangian is classically equivalent to (2.14). The metric components of $\gamma^{\alpha\beta}$ enter as Lagrange multipliers giving rise to the two constraints

$$Str\,\pi^2 + Str\big(\mathcal{A}_1^{(2)}\big)^2 = 0, \qquad Str\,\pi\,\mathcal{A}_1^{(2)} = 0. \tag{2.16}$$

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[4]It is also related to parity invariance of σ. Sending $\sigma \to -\sigma$ induces a sign change of κ.

Loosely speaking the solution of the first constraint give the gauge fixed string Hamiltonian while the second allow us to solve for one of the unphysical light-cone coordinates.

The auxiliary field π allows for a basis decomposition with respect to $\mathfrak{osp}(2,2|6)$,

$$\pi = \pi_+ \Sigma_+ + \pi_- \Sigma_- + \pi_t, \tag{2.17}$$

where

$$\pi_t = \begin{pmatrix} \pi_i^{(z)} \Gamma_i & 0 \\ 0 & \pi^{(y)} T_5 + \pi_i^{(\omega)} \tau_i + \bar{\pi}_i^{(\bar{\omega})} \bar{\tau}_i \end{pmatrix}$$

Note that from this basis decomposition, one see that $Str\,\pi\,\mathcal{A}^{(2)} = Str\,\pi\,\mathcal{A}^{even} = Str\,\pi\,\mathcal{A}$ which we used in (2.15).

One can think about the field π as the matrix version of a first order formalism. By introducing this field we effectively get rid of the worldsheet metric which make the process of bosonic gauge fixing considerably simpler. However, it is important to understand that the components of π does not directly correspond to the conjugate momentas of the bosonic fields. In order to obtain the physical Hamiltonian, one have to solve for these components and use the solutions in the Lagrangian (2.15). We will discuss this point in more detail in the next section.

2.3 Gauge fixing and field content

The Lagrangian (2.14) and (2.15) are invariant under two dimensional diffeomorphisms, Weyl scalings and fermionic kappa symmetry where the latter is a local worldsheet symmetry with odd transformation parameter.

The bosonic symmetries are used to fix a uniform light-cone gauge as

$$x^+ = \sigma^0 = \tau, \qquad p_+ = \text{Constant}, \tag{2.18}$$

which has the important consequence that the string coupling, g, becomes related to the length of the string, $g \sim L$ [53].

The model contains 24 real fermions whereas supersymmetry demands that the number of fermionic and bosonic excitations should be equal. At first glance, this looks like a problem since common lore has is that kappa symmetry removes half of the fermions, which in our case would leave us with to few fermions for supersymmetry to be manifest. However, as it turns out, the kappa symmetry for strings in $AdS_4 \times \mathbb{CP}_3$ is partial and only allows for eight real fermions to be removed [9]. Therefore the kappa fixed model has equal number of fermionic and bosonic excitations.

There are many ways to impose the kappa symmetry. In this paper we will use an especially convenient gauge introduced by Bykov which is compatible with the

bosonic part of the subgroup that commutes with the gauge-fixed string Hamiltonian[5] [27].

It can be shown that the light-cone Hamiltonian is proportional to $Str\,Q\,\Sigma_+$, where Q is the Noether charge associated with the global OSP(2,2|6) symmetry [51]. A specific symmetry generator can be expressed as a linear combination of Q traced over various basis element of $\mathfrak{osp}(2,2|6)$. The commutator between two charges, say $Q_1 = Str\,Q\,\mathcal{M}_1$ and $Q_1 = Str\,Q\,\mathcal{M}_2$, is given by

$$[Q_1, Q_2]_\pm \sim Str\,Q\,[\mathcal{M}_1, \mathcal{M}_2]_\pm, \tag{2.19}$$

where the $\pm$ is $+$ only if both charges are odd. It is easy to see that the subalgebra that commutes with the light-cone Hamiltonian is given by matrices of the form

$$\{\mathcal{M} \in \mathfrak{osp}(2,2|6)\,,\,[\Sigma_+, \mathcal{M}] = 0\} = \mathfrak{su}(2|2) \oplus \mathfrak{u}(1). \tag{2.20}$$

The bosonic part of this subalgebra is $\mathfrak{su}(2)_{AdS} \oplus \mathfrak{su}(2)_{CP} \oplus \mathfrak{u}(1)_{CP}$, where the subscript denotes which space the isometry originates from. In a matrix notation, elements in the bosonic subalgebra takes the form

$$\mathfrak{g}_B = \begin{pmatrix} \mathfrak{su}(2)_{AdS}|_{4\times4} & 0 & 0 \\ 0 & \mathfrak{u}(1)_{CP}|_{2\times2} & 0 \\ 0 & 0 & \mathfrak{su}(2)_{CP}|_{4\times4} \end{pmatrix}$$

A important fact is that all elements in $\mathfrak{g}_B$ also commute with Σ_-. This has the important consequence that these transformations only act on the transverse part of the group elment,

$$[\Sigma_\pm, \mathfrak{g}_B] = 0 \to [\mathfrak{g}_B, \Lambda(x^+, x^-)] = 0. \tag{2.21}$$

So, if we let $e^{\mathfrak{g}_B}$ act on G we find

$$e^{\mathfrak{g}_B} \cdot G = \Lambda(x^+, x^-)\,g_B\,f(\eta)\,g_B^{-1}\,g_B\,G_t\,g_B^{-1} \cdot g_c, \tag{2.22}$$

where g_c is a irrelevant compensating transformation from the stabilizer group, SO(1,3)×U(3). From this we see that the bosons and the fermions are in the adjoint representation of G^B=SU(2)×SU(2)×U(1).

As was explained in [27], a kappa gauge that transform covariantly under G^B can be constructed by first enforcing

$$\theta_{1,5} = i\,\theta_{1,4}, \quad \theta_{1,6} = i\,\theta_{1,3}, \quad \theta_{2,5} = i\,\theta_{2,4}, \quad \theta_{2,6} = i\,\theta_{2,3}, \tag{2.23}$$

which removes four complex fermions and thus leave us with a total of sixteen real ones as desired[6]. As it stands, the gauge (2.23), does not transform covariantly under

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[5]For another covariant kappa gauge, see [35].

[6]One can also think about the kappa gauge in the following way; if we anticommute a generic, non kappa gauge fixed odd matrix, with Σ_+, one find that the resulting object has the form of a kappa gauge fixed matrix. In one sense this can be seen as a defining property of the gauge. This is very similar to the kappa gauge imposed in [48] where the gauge fixing was defined through a commutation relation between a light-cone basis element and η.

the bosonic symmetries. However, if we augment the gauge with the following linear combinations of the spinor components[7]

$$\theta_{1,1} = \kappa^{+1} - \kappa_{+2}, \quad \theta_{1,2} = -i(\kappa^{+1} + \kappa_{-2}), \quad \theta_{2,1} = \kappa^{+2} + \kappa_{-1}, \quad \theta_{2,2} = -i(\kappa^{+2} - \kappa_{-1}),$$
$$\theta_{1,3} = \frac{1}{2}(s_{\dot{1}}^1 - s_{\dot{2}}^1), \quad \theta_{1,4} = -\frac{i}{2}(s_{\dot{1}}^1 + s_{\dot{2}}^1), \quad \theta_{2,3} = \frac{1}{2}(s_{\dot{1}}^2 - s_{\dot{2}}^2), \quad \theta_{2,4} = -\frac{i}{2}(s_{\dot{1}}^2 + s_{\dot{2}}^2), (2.24)$$

then the new variables transform under G^B as

$$\kappa^{+,a} \to e^{i\alpha} g_b^a \kappa^{+b}, \quad \kappa_{-a} \to e^{-i\alpha} g_a^b \kappa_{-b}, \quad s_{\dot{b}}^a \to g_b^a g_{\dot{b}}^{\dot{a}} s_{\dot{a}}^b, \tag{2.25}$$

where $g_b^a \in \mathrm{SU}(2)_{AdS}$, $g_{\dot{b}}^{\dot{a}} \in \mathrm{SU}(2)_{CP}$ and $e^{\pm i\phi} \in \mathrm{U}(1)$. Thus, in our notation, undotted indices correspond to the SU(2) from the AdS space and dotted ones correspond to the SU(2) from $\mathbb{CP}_3$. In this notation it becomes clear that we have two set of spinors, $\kappa^{\pm}$, with opposite U(1) charge transforming under the AdS SU(2)[8]. There is also a spinor, $s_{\dot{b}}^a$, uncharged under the U(1) but in a bifundamental representation of the two SU(2)'s.

We should also classify how the bosonic fields transform. Clearly, the z_i coordinates only transform under the SU(2) from the AdS space. The singlet y does not transform at all, neither under any SU(2) or the U(1). The only bosonic fields charged under the U(1) are the complex ω_i and $\bar{\omega}_i$ which also transform under the SU(2) of $\mathbb{CP}_3$. A convenient index notation is

$$\omega_i \to \omega_{\dot{a}}, \qquad \bar{\omega}_i \to \bar{\omega}^{\dot{a}}, \tag{2.26}$$

where lower index has the plus charge of the U(1) and vice versa.

Under conjugation, all indices changes place

$$(\kappa^{+a})^\dagger = \bar{\kappa}_{+a} = \epsilon_{ab} \bar{\kappa}^{+b}, \quad (\kappa_{-a})^\dagger = \bar{\kappa}^{-a} = \epsilon^{ab} \bar{\kappa}_{-b}, \tag{2.27}$$
$$(s_{\dot{b}}^a)^\dagger = \bar{s}_a^{\dot{b}} = \epsilon^{\dot{b}\dot{a}} \epsilon_{ab} \bar{s}_{\dot{a}}^b, \quad (\omega_{\dot{a}})^\dagger = \bar{\omega}^{\dot{a}} = \epsilon^{\dot{a}\dot{b}} \omega_{\dot{b}}, \quad \epsilon_{ab}\,\epsilon^{bc} = \delta_a^c, \quad \epsilon_{\dot{a}\dot{b}}\,\epsilon^{\dot{b}\dot{c}} = \delta_{\dot{a}}^{\dot{c}},$$

where we also introduced epsilon tensors to raise and lower indices, with the convention $\epsilon_{01} = 1 = -\epsilon^{01}$. It is convenient to let the $\pm$, denoting U(1) charge of the unconjugated spinors, travel with the SU(2) index. This imply that all lower $\pm$ have negative U(1) while upper have positive.

2.4 Light-cone Lagrangian and Hamiltonian

Having imposed the bosonic and fermionic gauges, we are in position to start extracting physical quantities from the string Lagrangian (2.15). The most natural

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[7]Note that the fermions denoted with $\kappa^\pm$ has no relation with the constant κ in front of the WZ term in the Lagrangian. Also note that the $\pm$ denotes U(1) charge and should not be confused as sign of the SU(2) index.

[8]The spinor transforming with negative U(1) is in the conjugate representation of the SU(2) from AdS, hence the lower index.

object to study is of course the string Hamiltonian. In the light-cone formalism it is given by minus the conjugate momenta of x^+ and it enters the Lagrangian in the natural way

$$\mathscr{L} = p_m \, \dot{x}^m + p_- + \text{Fermions}, \qquad m \in \{i, y, \dot{a}\}.$$

The Hamiltonian, $-p_-$, is a function of the physical fields and the auxiliary field π. The auxiliary field does not directly correspond to the momentum variables of the bosonic fields. Rather, each component of π can be expressed, and solved for, in terms of them. To extract the light-cone Hamiltonian in terms of physical fields we will proceed below as follows; for all but the π_- component, we use the conjugate momentas to solve for the components, that is

$$\frac{\delta \mathscr{L}}{\delta \dot{x}_M} = p_M = f(\text{fields} \neq \pi_M, \pi_M) \to \pi_M = \tilde{f}(\text{fields} \neq p_M, p_M), \qquad M \neq -.$$

Doing this for the transverse momenta shows that[9]

$$\pi_i^{(z)} = \frac{2i \, p_i^{(z)}}{4 + z_i^2}, \quad \pi^{(y)} = \frac{4 \, p_y}{8 + y^2 - \omega_{\dot{a}} \, \bar{\omega}^{\dot{a}}}, \tag{2.28}$$

$$\pi_{\dot{1}}^{(\omega)} = \frac{8 \, p_{\dot{1}} + \omega_{\dot{1}} \, \bar{\omega}^{\dot{2}} \, \pi_{\dot{2}}^{(\omega)}}{8 - \omega_{\dot{1}} \, \bar{\omega}^{\dot{1}} - \omega_{\dot{a}} \, \bar{\omega}^{\dot{a}}}, \quad \pi_{\dot{2}}^{(\omega)} = \frac{8 \, p_{\dot{2}} + \omega_{\dot{2}} \, \bar{\omega}^{\dot{1}} \, \pi_{\dot{1}}^{(\omega)}}{8 - \omega_{\dot{2}} \, \bar{\omega}^{\dot{2}} - \omega_{\dot{a}} \, \bar{\omega}^{\dot{a}}},$$

which more or less by definition satisfy

$$Str \, \pi \, G_t^{-1} \, \dot{G}_t = p_m \, \dot{x}^m, \tag{2.29}$$

where m runs over transverse indices.

The expressions for $\pi_\pm$ are considerably more complicated and for these components we will only present the corresponding matrix equations[10]. To obtain π_+ we solve for p_+ in a similar way as we did above, then use this solution in the quadratic constraint (2.16) to solve for π_-,

$$\pi_+ = -\pi_- \frac{Str \, \Sigma_- \, G_-}{Str \, \Sigma_+ \, G_-} + \frac{1}{Str \, \Sigma_+ \, G_-} \Big(\mathbb{p}_+ - Str \, \pi_t \, G_- \Big), \tag{2.30}$$

$$\pi_- = \frac{\mathbb{p}_+ - Str \, \pi_t \, G_-}{2 \, Str \, \Sigma_- \, G_-} \left\{ 1 \pm \sqrt{1 - \frac{\left(Str \, \Sigma_- \, G_- \right) \left(Str \, \Sigma_+ \, G_- \right) \left(Str \, \pi_t^2 + Str \left(\mathcal{A}_1^2 \right)^2 \right)}{4 \left(\mathbb{p}_+ - Str \, \pi_t \, G_- \right)^2}} \right\}$$

$$= \frac{\left(Str \, \Sigma_+ \, G_- \right) \left(Str \, \pi_t^2 + Str \left(\mathcal{A}_1^2 \right)^2 \right)}{16 \left(\mathbb{p}_+ - Str \, \pi_t \, G_- \right)} + \dots$$

---

[9]As can be seen, the complex components mix within each other and one might be tempted to shift the fields so this complication disappears. However, as it turns out this mixing enters only at quartic order in number of fields so for the upcoming perturbative analysis this mixing is irrelevant.

[10]However, their quadratic part is needed to determine the upcoming fermionic shift, so these parts we present in (C.2).

where we introduced the short hand notation G_- for the even part of

$$\frac{i}{2}\,G_t^{-1}\left(f^{-1}(\eta)\,\Sigma_-\,f(\eta)\right)G_t$$

and $\mathbb{p}_+$ is

$$\mathbb{p}_+ = p_+ - p_+^{WZ} = \tag{2.31}$$
$$p_+ - \kappa\frac{i}{2}\,Str\left\{G_t^{-1}\left(\frac{i}{2}\sqrt{1+\eta^2}\,\Sigma_-\,\eta - \frac{i}{2}\eta\,\Sigma_-\,\sqrt{1+\eta^2}\right)G_t\,\Upsilon\,\mathcal{A}_1^{Odd}\,\Upsilon^{-1}\right\},$$

where the last part is the contribution to p_+ coming from the WZ term and $\mathcal{A}^{Odd} = \mathcal{A}^{(1)} + \mathcal{A}^{(3)}$.

The light-cone Hamiltonian is given by

$$-\mathcal{H} = p_- = \frac{\delta\mathcal{L}}{\delta\dot{x}^+} = \tag{2.32}$$
$$\frac{i}{2}\,Str\,\pi\,G_t^{-1}\left(\Sigma_+ - \eta\,\Sigma_+\,\eta + \sqrt{1+\eta^2}\,\Sigma_+\,\sqrt{1+\eta^2}\right)G_t$$
$$-\kappa\frac{i}{2}\,Str\left\{G_t^{-1}\left(\frac{i}{2}\sqrt{1+\eta^2}\,\Sigma_+\,\eta - \frac{i}{2}\eta\,\Sigma_+\,\sqrt{1+\eta^2}\right)G_t\times\right.$$
$$\left.\Upsilon\,G_t^{-1}\left(\left(\frac{i}{2}\sqrt{1+\eta^2}\,\Sigma_-\,\eta - \frac{i}{2}\eta\,\Sigma_-\,\sqrt{1+\eta^2}\right)x'^- + \sqrt{1+\eta^2}\,\eta' - \eta\,\partial_1\sqrt{1+\eta^2}\right)G_t\,\Upsilon^{-1}\right\}.$$

As it stands, the expression above is very involved. To be able to extract anything useful from it one need to consider various simplifying limits, which will be the main topic of the next section.

Combining everything we have so far, we can write the string Lagrangian as

$$\mathcal{L} = \tag{2.33}$$
$$p_+\,\dot{x}^- + p_m\,\dot{x}^m + p_- + Str\,\pi\,G_t^{-1}\left(-\eta\,\dot{\eta} + \sqrt{1+\eta^2}\,\partial_0\sqrt{1+\eta^2}\right)G_t$$
$$+\frac{i}{2}\kappa\,Str\,G_t^{-1}\left(\sqrt{1+\eta^2}\,\partial_0\eta - \eta\,\partial_0\sqrt{1+\eta^2}\right)G_t\,\Upsilon\,\mathcal{A}_1^{Odd}\,\Upsilon^{-1}.$$

Together with the solutions for π and the expression for p_- in (2.32) this is the *exact* gauge fixed string Lagrangian for the AdS$_4 \times \mathbb{CP}_3$ superstring. It will be the starting point for a perturbative analysis in the next section. However, it should be clear that the terms involving time derivatives of the fermions will have terms beyond quadratic order. This severely complicates the quantization procedure since we get a very involved Poisson structure for the fermionic quantities, see [47] for an example. Luckily, one can side pass this complication by performing a shift of the fermions in such a way that the higher order kinetic terms vanish. This has the advantage of a canonical Poisson structure but with the cost of additional terms in the light-cone Hamiltonian.

3. Strong coupling expansion

To be able to extract anything useful from (2.32) we have to consider some sort of perturbative expansion. The standard way to proceed is to boost, spin or deform the string in some way or another. In this paper we will expand around a point like string configuration moving on a null geodesic. Or equivalently, a plane wave expansion [56]. In practise the limit boils down to the following expansion scheme[11]

$$g \to \infty, \qquad x_m \to \frac{x_m}{\sqrt{g}}, \qquad p_m \to \frac{p_m}{\sqrt{g}}, \qquad \eta \to \frac{\eta}{\sqrt{g}}, \qquad (3.1)$$

which becomes an expansion in number of fields [48].

An important physical consequence of the above limit is that the string length, which was proportional to g, becomes infinite. The worldsheet of a closed string has the topology of a cylinder so taking the $g \to \infty$ limit means that the string decompactifies. It becomes a infinite plane. In terms of string Bethe equations and asymptotic configurations, this fact has far reaching consequences, see [36] and references therein.

3.1 Leading order

It is a good idea to start out the perturbative analysis by fixing some of the constants we encountered so far. First of all, from now on we will fix[12]

$$p_+ = 1 \qquad \kappa = 1. \qquad (3.2)$$

What we choose to do with our parameter space is of course arbitrary and the physics we want to extract is totally independent of numerical conventions. However, the choices above are very convenient in terms of notation. Having factors of κ and p_+ in the expressions makes things which are, and especially will become, complicated more involved than necessary.

It is also desirable to have the Lagrangian in such a form that the field expansions becomes as simple as possible. To achieve this we rescale the string length parameter as $\sigma \to 2\sigma$ and send[13] $\eta \to i\eta$. Taking this into consideration, and taking the limit (3.1) of (2.33) gives the leading order quadratic Lagrangian

$$\frac{1}{2}\mathscr{L} = p_i\,\dot{z}_i + p_y\,\dot{y} + \dot{w}_{\dot{a}}\,\bar{p}^{\dot{a}} + \dot{\bar{\omega}}^{\dot{a}}\,p_{\dot{a}} + i\bar{s}^{\dot{b}}_a\,\dot{s}^a_{\dot{b}} + i\bar{\kappa}_{+\,a}\,\dot{\kappa}^{+\,a} + i\bar{\kappa}^{-\,a}\,\dot{\kappa}_{-\,a} \qquad (3.3)$$

$$-p_i^2 - 4\bar{p}^{\dot{a}}\,p_{\dot{a}} - p_y^2 - \frac{1}{4}\Big(y^2 + z_i^2 + \frac{1}{4}\bar{\omega}^{\dot{a}}\,\omega_{\dot{a}}\Big) - \frac{1}{4}\Big(z_i'^2 + y'^2 + \bar{\omega}'^{\dot{a}}\,\omega'_{\dot{a}}\Big)$$

$$-\bar{s}^{\dot{b}}_a\,s^a_{\dot{b}} - \frac{1}{2}\Big(\bar{\kappa}_{+\,a}\,\kappa^{+\,a} + \bar{\kappa}^{-\,a}\,\kappa_{-\,a}\Big) - i\Big(\kappa_{-\,a}\,\kappa'^{+\,a} + \bar{\kappa}_{+\,a}\,\bar{\kappa}'^{-\,a}\Big)$$

$$-\frac{i}{2}\Big(s^a_{\dot{b}}\,(s')^{\dot{b}}_a + \bar{s}^{\dot{b}}_a\,(\bar{s}')^a_{\dot{b}}\Big).$$

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[11]This is essentially the same expansion scheme as for the string in [48] and [26] with the effective BMN coupling $\tilde{\lambda}$ put to unity.

[12]Once again we stress that the κ here has nothing to do with the two fermions $\kappa^{\pm}$.

[13]This is equivalent to defining the fermionic part of the group element as $f(\eta) = \sqrt{1 - \eta^2} + i\,\eta$.

From this we find that the fields come in heavy and light multiplets,

$$\mathbf{M} = 1; \qquad \{s_b^a,\, z_i,\, y\} \qquad \mathbf{M} = \frac{1}{2}; \qquad \{\kappa^{+a},\, \kappa_{-a},\, \omega_{\dot{a}},\, \bar{\omega}^{\dot{a}}\}.$$

This $4_{\frac{1}{2}} + 4_1$ split of the masses is a novel feature for the $\mathrm{AdS}_4 \times \mathbb{CP}_3$ string. In the last section of this paper we will calculate loop corrections to propagators for the massive modes. There it will be argued that the heavy excitations can be viewed as composite states of light modes. For now though we view them as single excitations.

Note that we all through out the paper work with phase space variables. The gauge fixing procedure is vastly simplified through the use of the auxiliary π field since it allowed us to eliminate the dependence of the worldsheet metric. The auxiliary field is expressed in terms of the two unknown $\pi_\pm$ components and the transverse momentum variables. If we desired, we could after the gauge fixing procedure is completed, express the momentum variables in terms of velocities resulting in a different, but completely equivalent, formulation of the theory, see [5] and [7]. However, as in for example [48] and [26], we find it convenient to stick with the phase space formulation. Also, the parameterization of the group element that we use is especially suitable for a Hamiltonian analysis since the transverse coordinates of the auxilitary field π, in (2.28), do not depend on any fermionic quantities.

We can tidy up the notation a bit further by making the quadratic 2-d Lorentz symmetry manifest. First we introduce, $\gamma^0 = \sigma_3$ and $\gamma^1 = -i\sigma_2$, which obeys $\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}$ with $(+, -)$ convention. We then combine the fermions into two spinors as

$$\Psi = \begin{pmatrix} \kappa^{+a} \\ \bar{\kappa}^{-a} \end{pmatrix}, \quad \bar{\Psi} = \Psi^\dagger \gamma^0 = \begin{pmatrix} \bar{\kappa}_{+a}, & -\kappa_{-a} \end{pmatrix}, \quad \chi = \begin{pmatrix} s_b^a \\ \bar{s}_b^a \end{pmatrix}, \quad \bar{\chi} = \chi^\dagger \gamma^0 = \begin{pmatrix} \bar{s}_a^b, & -s_a^b \end{pmatrix}.$$

Then the quadratic Lagrangian can be written as

$$\frac{1}{2}\mathcal{L} = p_i\, \dot{z}_i + p_y\, \dot{y} + \dot{w}_{\dot{a}}\, \bar{p}^{\dot{a}} + \dot{\bar{\omega}}^{\dot{a}}\, p_{\dot{a}} \tag{3.4}$$
$$-p_i^2 - 4\bar{p}^{\dot{a}}\, p_{\dot{a}} - p_y^2 - \frac{1}{4}\Big(y^2 + z_i^2 + \frac{1}{4}\bar{\omega}^{\dot{a}}\, \omega_{\dot{a}}\Big) - \frac{1}{4}\Big(z_i'^2 + y'^2 + \bar{\omega}'^{\dot{a}}\, \omega_{\dot{a}}'\Big)$$
$$+i\bar{\Psi}\, \gamma^\alpha\, \partial_\alpha\, \Psi + \frac{i}{2}\bar{\chi}\, \gamma^\alpha\, \partial_\alpha\, \chi - \frac{1}{2}\bar{\Psi}\, \Psi - \frac{1}{2}\bar{\chi}\, \chi.$$

Anticipating the quantization procedure we expand the fields in Fourier coefficients

as

$$\omega_{\dot{a}} = \frac{1}{\sqrt{2\,\pi}} \int dp \, \frac{1}{\sqrt{\omega_p}} \Big(a^{\dot{a}}\, e^{ip\sigma} + \bar{b}^{\dot{a}}\, e^{-ip\sigma} \Big), \quad p_{\dot{a}} = \frac{i}{\sqrt{2\,\pi}} \int dp \, \frac{\sqrt{\omega_p}}{4} \Big(\bar{b}^{\dot{a}}\, e^{-ip\sigma} - a^{\dot{a}}\, e^{ip\sigma} \Big),$$

$$y = \frac{1}{\sqrt{2\,\pi}} \int dp \, \frac{1}{\sqrt{2\Omega_p}} \Big(y\, e^{ip\sigma} + \bar{y}\, e^{-ip\sigma} \Big), \quad p_y = \frac{1}{2} \frac{i}{\sqrt{2\,\pi}} \int dp \, \sqrt{\frac{\Omega_p}{2}} \Big(\bar{y}\, e^{-ip\sigma} - y\, e^{ip\sigma} \Big),$$

$$z_i = \frac{1}{\sqrt{2\,\pi}} \int dp \, \frac{1}{\sqrt{2\Omega_p}} \Big(z_i\, e^{ip\sigma} + \bar{z}_i\, e^{-ip\sigma} \Big), \quad p_i = \frac{1}{2} \frac{i}{\sqrt{2\,\pi}} \int dp \, \sqrt{\frac{\Omega_p}{2}} \Big(\bar{z}_i\, e^{-ip\sigma} - z_i\, e^{ip\sigma} \Big),$$

$$s_b^a = \frac{1}{\sqrt{2\pi}} \int dp \frac{1}{\sqrt{2\Omega_p}} \Big(F_p\, \chi_b^a\, e^{ip\sigma} - H_p\, \bar{\chi}_b^a\, e^{-ip\sigma} \Big),$$

$$\kappa^{+\,a} = \frac{1}{\sqrt{2\pi}} \int dp \frac{1}{\sqrt{2\omega_p}} \Big(f_p\, c^a\, e^{ip\sigma} - h_p\, \bar{d}^a\, e^{-ip\sigma} \Big),$$

$$\kappa_{-\,a} = \frac{1}{\sqrt{2\pi}} \int dp \frac{1}{\sqrt{2\omega_p}} \Big(f_p\, d_a\, e^{ip\sigma} - h_p\, \bar{c}_a\, e^{-ip\sigma} \Big),$$

and obvious ones for conjugated fields. The frequencies and the fermionic wave functions are given by,

$$\omega_p = \sqrt{\frac{1}{4} + p^2}, \quad f_p = \sqrt{\frac{\omega_p + \frac{1}{2}}{2}}, \quad h_p = \frac{p}{2 f_p}, \tag{3.5}$$

$$\Omega_p = \sqrt{1 + p^2}, \quad F_p = \sqrt{\frac{\Omega_p + 1}{2}}, \quad H_p = \frac{p}{2 F_p},$$

where the wave functions satisfy the following important identities,

$$f_p^2 + h_p^2 = \omega_p, \quad f_p^2 - h_p^2 = \frac{1}{2}, \quad F_p^2 + H_p^2 = \Omega_p, \quad F_p^2 - H_p^2 = 1.$$

If we now plug the field expansion into (3.3) and integrate over σ, we find

$$L = \int dp \, \Big(i \big(\bar{b}^{\dot{b}}\, \dot{b}_{\dot{b}} + \bar{a}_{\dot{b}}\, \dot{a}^{\dot{b}} + \bar{y}\, \dot{y} + \bar{z}_i\, \dot{z}_i + \bar{\chi}_a^b\, \dot{\chi}_b^a + \bar{c}_a\, \dot{c}^a + \bar{d}^a\, \dot{d}_a \big) \tag{3.6}$$

$$- \omega_p \big(\bar{b}^{\dot{b}}\, b_{\dot{b}} + \bar{a}_{\dot{b}}\, a^{\dot{b}} + \bar{c}_{\dot{b}}\, c^a + \bar{d}^a\, d_a \big) - \Omega_p \big(\bar{y}\, y + \bar{z}_i\, z_i + \bar{\chi}_a^b\, \chi_b^a \big) \Big).$$

We also need to consider the second constraint in (2.16) which give rise to

$$\mathcal{V} = \int dp \, p \Big(\bar{b}^{\dot{b}}\, b_{\dot{b}} + \bar{a}_{\dot{b}}\, a^{\dot{b}} + \bar{y}\, y + \bar{z}_i\, z_i + \bar{c}_{\dot{b}}\, c^a + \bar{d}^a\, d_a + \bar{\chi}_a^{\dot{b}}\, \chi_b^a \Big). \tag{3.7}$$

Which is the so called level matching constraint enforcing that the sum of all mode numbers has to vanish for physical states. In the quantum theory this will be promoted to an operator whose action on a physical state should project to zero.

Promoting the oscillators to operators is now down by imposing the equal time (anti)commutators

$$[a(p,\tau)^{\dot{a}}, \bar{a}(p',\tau)_{\dot{b}}] = 2\pi\, \delta^{\dot{a}}_{\dot{b}}\, \delta(p-p'), \quad [b(p,\tau)_{\dot{a}}, \bar{b}(p',\tau)^{\dot{b}}] = 2\pi\, \delta^{\dot{b}}_{\dot{a}}\, \delta(p-p') \quad (3.8)$$

$$[y(p,\tau), \bar{y}(p',\tau)] = 2\pi\, \delta(p-p'), \quad [z_i(p,\tau), \bar{z}_j(p',\tau)] = 2\pi\, \delta_{ij}\delta(p-p'),$$

$$\{c^a(p,\tau), \bar{c}_b(p',\tau)\} = \{d_b(p,\tau), \bar{d}^a(p',\tau)\} = 2\pi\, \delta^a_b\, \delta(p-p'),$$

$$\{\chi^a_{\dot{a}}(p,\tau), \bar{\chi}^{\dot{b}}_b(p',\tau)\} = 2\pi\, \delta^a_b\, \delta^{\dot{a}}_{\dot{b}}\, \delta(p-p').$$

With this we have established the quadratic Lagrangian, including field expansions and commutation relations. We would now like to proceed to the higher order contributions from (2.32). However, before extracting the sub leading terms in the light-cone Hamiltonian, we have to take care of the higher order kinetic fermions. If these were to be included then the anti commutation relations in (3.8) would receive higher order corrections. In the next section we will describe how this complication can (partially) be avoided by a appropriate shift of the fermions.

3.2 Canonical fermions

The focus of this section be will the piece of (2.33) that contains kinetic fermionic terms,

$$\mathcal{L}^{\eta}_{Kinetic} = \tag{3.9}$$

$$\frac{1}{2} Str\, \pi\, G_t^{-1} \Big([\dot{\eta}, \eta] + \frac{1}{4}[\eta^2, \{\dot{\eta}, \eta\}] \Big) G_t$$

$$-\frac{i}{2}\kappa\, Str\, G_t^{-1} \Big(\dot{\eta} - \frac{1}{2}\eta\, \dot{\eta}\, \eta \Big)\, G_t\, \Upsilon\, G_t^{-1} \Big(\frac{i}{2}\, [\Sigma_-, \eta]\, x'^- + \eta' - \frac{1}{2}\eta\, \eta'\, \eta \Big) G_t\, \Upsilon^{-1} + \mathcal{O}(\eta^6),$$

from which it is clear that the anti commutation relations in (3.8) will receive higher order contributions. In principle this is not a fundamental problem and it can be solved explicitly by a careful analysis of the Poisson structure, see for example [47]. However, from a calculational point of view, it is rather cumbersome to deal with non trivial commutation relations. For that reason we will try to avoid the problem by performing a shift of the fermionic coordinates[14].

By using the cyclicity of the super trace and the form of π_+, we can write[15]

$$\mathcal{L}^{\eta}_{Kinetic} = \frac{i}{4}\, Str\, \Sigma_+\, \dot{\eta}\, \eta + Str\, \dot{\eta}\, \widetilde{\Phi}(x_m, p_m, \eta), \tag{3.10}$$

where $\widetilde{\Phi}(x_m, p_m, \eta)$ is a complicated fermionic matrix, presented in (C.1), that can be deduced from (3.9). It starts at quadratic order in number of fields and for the analysis at hand we have to know it up to cubic order[16]

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[14]For a similar but much simpler discussion, see [48].

[15]π_+ is the only component of the auxiliary field which has a constant leading order term.

[16]The observant reader might notice that (3.9) also has a second quadratic piece $\sim Str\, \dot{\eta}\, \Upsilon\, \eta'\, \Upsilon^{-1}$. This term is, however, a total derivative and can be neglected.

We will now show that most of the higher order terms can be removed by shifting the fermions in an appropriate way. First we introduce a, so far arbitrary, function $\Phi(x_m, p_m, \eta)$. Since we are to expand the Hamiltonian up to quartic order, we need this function to third order in number of fields. To simplify the notations we split up $\Phi(x_m, p_m, \eta)$ in number of fields and leave the bosonic dependence implicit, $\Phi(x_m, p_m, \eta) = \Phi_2(\eta) + \Phi_3(\eta)$. The idea is now to shift the fermionic matrix as

$$\eta \rightarrow \eta + \Phi(\eta). \tag{3.11}$$

Performing the shift in (3.10) and writing, $\widetilde{\Phi}(x_m, p_m, \eta) = \widetilde{\Phi}_2(\eta) + \widetilde{\Phi}_3(\eta)$, we find

$$\mathscr{L}^\eta_{Kinetic} = \tag{3.12}$$
$$\frac{i}{4} Str\, \Sigma_+ \dot{\eta}\,\eta + Str\, \dot{\eta}\left(\widetilde{\Phi}_2(\eta) + \widetilde{\Phi}_3(\eta)\right) + \frac{i}{4} Str\, \dot{\eta}[\Phi_2(\eta) + \Phi_3(\eta), \Sigma_+]$$
$$+ Str\, \dot{\eta}\,\widetilde{\Phi}_2(\eta \rightarrow \Phi_2) + Str\, \dot{\Phi}_2(\eta)\,\widetilde{\Phi}_2(\eta) + \frac{i}{4} Str\, \Sigma_+\, \dot{\Phi}_2(\eta)\, \Phi_2(\eta),$$

where $\widetilde{\Phi}_2(\eta \rightarrow \Phi_2)$ is a cubic contribution from $\widetilde{\Phi}$ with Φ_2 as argument.

To proceed, we need to find the form of Φ. We do this by recalling that a general kappa gauge fixed fermionic element, which we again call η, can be written as a commutator, $\eta = [\Sigma_+, \chi]$ for some arbitrary, non kappa gauge fixed, fermionic matrix χ. This means that a term of the form $Str\, \dot{\eta}\,\widetilde{\Phi}$, for arbitrary fermionic $\widetilde{\Phi}$, can be written $Str\, \dot{\chi}\,[\Sigma_+, \widetilde{\Phi}]$. This imply that for Φ to remove the higher order terms, it should satisfy the matrix equation

$$[\Sigma_+, [\Phi, \Sigma_+]] + [\Sigma_+, \widetilde{\Phi}] = 0. \tag{3.13}$$

Some trial and error shows that a solution for Φ in terms of $\widetilde{\Phi}$ is

$$\Phi = \begin{pmatrix} \mathbb{1}_{6\times 6} & 0 \\ 0 & \frac{1}{4}\mathbb{1}_{4\times 4} \end{pmatrix} [\Sigma_+, \widetilde{\Phi}] \begin{pmatrix} \mathbb{1}_{6\times 6} & 0 \\ 0 & \frac{1}{4}\mathbb{1}_{4\times 4} \end{pmatrix} = \Gamma\,[\Sigma_+, \widetilde{\Phi}]\,\Gamma,$$

which allows us to remove the $Str\, \dot{\eta}\,\widetilde{\Phi}$ terms in (3.12) by choosing,

$$\Phi = -4i\,\Gamma\,[\Sigma_+, \widetilde{\Phi}_2 + \widetilde{\Phi}_2(\eta \rightarrow \Phi_2) + \widetilde{\Phi}_3]\,\Gamma. \tag{3.14}$$

This leaves us with

$$\mathscr{L}^\eta_{Kin} = \frac{i}{4} Str\, \Sigma_+ \dot{\eta}\,\eta + Str\, \dot{\Phi}_2\,\widetilde{\Phi}_2 + \frac{i}{4} Str\, \Sigma_+\, \dot{\Phi}_2\, \Phi_2, \tag{3.15}$$

which can be rewritten using (3.14) to

$$\mathscr{L}^\eta_{Kin} = \frac{i}{4} Str\, \Sigma_+ \dot{\eta}\,\eta + \frac{1}{2} Str\, \dot{\Phi}_2\,\widetilde{\Phi}_2. \tag{3.16}$$

The last expression is unfortunately rather involved. It is of quartic order in number of fields and introduce additional time derivatives of the bosonic fields since

$$\widetilde{\Phi}_2 = \frac{1}{2}\Big(\frac{i}{4}[\eta, [G_t^1, \Sigma_+]] + [\eta, \pi_t^1]\Big) - \frac{i}{2}\big([G_t^1, \Upsilon]\, \eta'\, \Upsilon^{-1} + \Upsilon\, \eta'\, [G_t^1, \Upsilon^{-1}]\big),$$

where G_t^1 and π_t^1 are the pieces of G_t and π_t linear in fields. To remove the additional fermionic kinetic terms induced by the shift, one needs to isolate the $\dot{\eta}$ terms from (3.16) and introduce a second shift, say $\hat{\Phi}_3$, with the property $\frac{i}{4}Str\, \dot{\eta}[\hat{\Phi}_3, \Sigma_+] = -\frac{1}{2}Str\, \Phi_2\, \widetilde{\Phi}_2|_{\dot{\eta}}$, where the notation is meant to imply the $\dot{\eta}$ dependent part of $Str\, \Phi_2\, \widetilde{\Phi}_2$. However, this means that the $\dot{\eta}$ independent part contains time derivatives of the bosonic fields, so we find corrections to the transverse part of π in (2.28). Needless to say, this analysis becomes rather involved. Not only will the additional fermionic shift, $\hat{\Phi}_3$, complicate things further, but the additional momentum terms also give rise to complications since they will have a quadratic fermionic dependence[17].

We will tackle this problem by simply ignoring it. Or, to be more precise, we assume that the $\hat{\Phi}_3$ shift is performed but do not determine the form of it, nor the additional momentum terms, allowing us to maintain the canonical Poisson structure for the fermions. The reason we can do this is because $Str\, \Phi_2\, \widetilde{\Phi}_2$ contains two fermions and two bosons, which implies that all additional terms, both from the shift and from π_t, will end up in the mixing part of the shifted Hamiltonian, $\mathcal{H}_{BF}$. This is acceptable since this part is not needed for the upcoming analysis.

However, a nice feature of the shift is that the x'^- dependence will cancel between the shifted and the original quartic Hamiltonian[18]. Another nice consequence of the shift is that it removes all fermionic non σ derivative terms from the relevant parts of the Hamiltonian. This is important since the point particle dynamics should be fully encoded in the quadratic fluctuations.

To summarize what we have done; We introduced a fermionic shift Φ, which can be expressed in terms of $\widetilde{\Phi}$, with the property that it removes all higher order fermionic derivative terms. However, due to the presence of cubic terms in the Lagrangian, the shift adds a 'self interaction' term of the form $Str\, \Phi_2\, \widetilde{\Phi}_2$. This term is not only complicated, but it also alters the transverse part of the auxiliary field π. Instead of determining this term explicitly, we simply assume the shift is performed, which guarantees a canonical Poisson structure. This is equivalent to put $Str\, \Phi_2\, \widetilde{\Phi}_2$ to zero by hand and accept that we can not determine the mixing part, $\mathcal{H}_{BF}$, of the shifted Hamiltonian. It is a bit surprising that the fermions are of such a complicated nature. For the AdS$_5 \times$S^5 string the corresponding shift actually

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[17]One could try to change the form of the OSP$(2, 2|6)$ group element as $G = \Lambda\, G_t\, f(\eta)$ which simplifies the fermionic kinetic term with the price of fermionic dependence in the bosonic conjugate momentas from start. However, pushing through with the analysis one finds that in the end the complications are more or less the same and the fermionic shift is still very involved.

[18]This is also true for the shifted $\mathcal{H}_{BF}$ part. The additional contributions from the complicated $Str\, \Phi_2\, \widetilde{\Phi}_2$ does introduce any additional x^- terms.

simplified the resulting theory, while here it has the opposite effect. Perhaps it is related to the coset construction we use which is not as rigorous as the AdS_5 string, see [49] and [50] for a related discussion.

What we can determine though is the shifted part of the Hamiltonian containing only bosons and fermions. This we will do in the next section. In the appendix we also present the full unshifted Hamiltonian, which together with the full form of the fermionic shift allows one to determine the shifted mixing Hamiltonian.

3.3 Higher order Hamiltonian

Having established the relevant form of the fermionic shift we are now in position to derive the Hamiltonian (2.32) to quartic order in fields. The way to do this is a straight forward, albeit somewhat tedious, multi step process. First we use the solution for π in (2.32), impose the shift (3.11) and expand to quartic order. It should be obvious that due to the complexity of both the Hamiltonian and the shift, it is very desirable to use some sort of computer program that can handle symbolic manipulations[19].

Pushing through with the calculation one find that the Hamiltonian has cubic next to leading order terms. This is another novel feature compared to the $AdS_5 \times S^5$ string which subleading terms start at quartic order.

Before we present our findings we would like to introduce yet another convenient notation,

$$Z_b^a = \sum_i z_i \sigma_{i,b}^a, \qquad Z^2 = \frac{1}{2} Tr \, Z_b^a \, Z_c^b = \sum_i z_i^2 \tag{3.17}$$

$$P_{z,b}^a = \sum_i p_i \sigma_{i,b}^a, \qquad P_z^2 = \frac{1}{2} Tr \, P_{z,b}^a \, P_{z,c}^b = \sum_i p_i^2,$$

where the Pauli matrices transform as $\sigma \rightarrow g \, \sigma \, g^t$ under the AdS SU(2).

With all this, we are now in position to extract the full Hamiltonian. Starting out with the subleading cubic part, we find

$$\sqrt{g} \, \mathcal{H}_3 = \tag{3.18}$$
$$\left(\bar{\Psi}_a \Psi^b \right)' Z_b^a + i \left(\bar{\Psi} \, \gamma^1 \, \Psi' - \bar{\Psi}' \gamma^1 \, \Psi \right)_a^b (Z')_b^a - 2i \left(\bar{\Psi}' \gamma^0 \, \Psi - \bar{\Psi} \, \gamma^0 \, \Psi' \right)_a^b P_{z,b}^a$$
$$+ 2 \left(\left(\bar{\chi}_{a\beta} \, \gamma^1 \, \Psi'^a - \bar{\chi}'_{ab} \, \gamma^1 \, \Psi^a \right) \bar{p}^\beta + \left(\bar{\Psi}_a \gamma^1 \, \chi'^{a\dot{b}} - \bar{\Psi}'_a \gamma^1 \, \chi^{a\dot{b}} \right) p_{\dot{b}} \right) + \frac{i}{4} \left(\left(\bar{\chi}_{a\dot{b}} \, \gamma^1 \, \gamma^0 \, \Psi^a \right)' \bar{\omega}^{\dot{b}} \right.$$
$$\left. + \left(\bar{\Psi}_a \gamma^0 \gamma^1 \, \chi^{a\dot{b}} \right)' \omega_{\dot{b}} \right) + \frac{1}{2} \left(\bar{\chi}_{a\dot{b}} \gamma^0 \, \Psi'^a - \bar{\chi}'_{a\dot{b}} \gamma^0 \, \Psi^a \right) \bar{\omega}'^{\dot{b}} + \frac{1}{2} \left(\bar{\Psi}_a \gamma^0 \chi'^{a\dot{b}} - \bar{\Psi}'_a \gamma^0 \chi^{a\dot{b}} \right) \omega'_{\dot{b}}$$
$$+ i \, y \left(\bar{p}^{\dot{b}} \, \omega_{\dot{b}} - p_{\dot{b}} \, \bar{\omega}^{\dot{b}} \right).$$

A nice feature of the coordinate system we use is that the massive singlet do not mix with any of the fermionic coordinates. Let us also remark that the fermionic shift (3.11) induces additional terms already here in the cubic Hamiltonian.

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[19]For this paper we made use of Mathematica version 7 together with the package [57].

We will split up the quartic Hamiltonian according to its bosonic / fermionic field content $g\,\mathcal{H}_4 = \mathcal{H}_{BB} + \mathcal{H}_{BF} + \mathcal{H}_{FF}$. For the pure bosonic contribution, we find

$$\frac{g}{2}\,\mathcal{H}_{BB} = \tag{3.19}$$

$$\frac{1}{4}Z^2\,Z'^2 - \frac{3}{4}\,p_y^2\,y^2 + \frac{1}{16}y^4 - \frac{1}{16}y^2\,y'^2 - \frac{1}{16}\bar{\omega}^{\dot{a}}\,\bar{\omega}'^{\dot{b}}\,\omega_{\dot{b}}\,\omega'_{\dot{a}} - \frac{3}{32}\bar{\omega}^{\dot{a}}\,\bar{\omega}'^{\dot{b}}\,\omega_{\dot{a}}\,\omega'_{\dot{b}}$$

$$-\frac{1}{128}\bar{\omega}^{\dot{a}}\,\bar{\omega}^{\dot{b}}\,\omega_{\dot{a}}\,\omega_{\dot{b}} + \frac{1}{2}\bar{p}^{\dot{a}}\,\bar{\omega}^{\dot{b}}\,p_{\dot{a}}\,\omega_{\dot{b}} + \bar{p}^{\dot{a}}\,\bar{\omega}^{\dot{b}}\,p_{\dot{b}}\,\omega_{\dot{a}} - \frac{1}{8}\bar{\omega}'^{\dot{a}}\,\omega'_{\dot{a}}\,y^2 - \frac{3}{32}\bar{\omega}^{\dot{a}}\,\omega_{\dot{a}}\,y'^2$$

$$-2\,\bar{p}^{\dot{a}}\,p_{\dot{a}}\,y^2 + \frac{1}{8}p_y^2\,\bar{\omega}^{\dot{a}}\,\omega_{\dot{a}} - \frac{1}{2}y^2\,P_z^2 - \frac{1}{8}\bar{\omega}^{\dot{a}}\,\omega_{\dot{a}}\,P_z^2 + 2\,\bar{p}^{\dot{a}}\,p_{\dot{a}}\,Z^2 + \frac{1}{8}\bar{\omega}'^{\dot{a}}\,\omega'_{\dot{a}}\,Z^2$$

$$-\frac{1}{32}\,\bar{\omega}^{\dot{a}}\,\omega_{\dot{a}}\,Z'^2 + \frac{1}{8}y'^2\,Z^2 + \frac{1}{2}p_y^2\,Z^2 - \frac{1}{8}y^2\,Z'^2,$$

which, for another more complicated coordinate system, was first calculated in [26].

Next we turn to the purely fermionic part which is given by[20],

$$g\,\mathcal{H}_{FF} = -i\big(\kappa_{-a}\,\bar{\kappa}_{+b}\,\kappa^{+a}\,\kappa'^{+b} + \kappa_{-a}\,\kappa'_{-b}\,\kappa^{+a}\,\bar{\kappa}^{-b}\big) - \frac{i}{2}\big(\kappa_{-a}\,\bar{\kappa}_{+b}\,\kappa'^{+a}\,\kappa^{+b} \tag{3.20}$$

$$+\kappa_{-a}\,\kappa'_{-b}\,\bar{\kappa}^{-a}\,\kappa^{+b} + \kappa_{-a}\,\bar{\kappa}'_{+b}\,\bar{\kappa}^{-a}\,\bar{\kappa}^{-b} + \bar{\kappa}_{+a}\,\bar{\kappa}_{+b}\,\kappa^{+a}\,\bar{\kappa}'^{-b}\big) + \big(\kappa_{-a}\,\bar{\kappa}_{+b}\,\bar{\kappa}'^{-a}\,\kappa'^{+b}$$

$$+\kappa_{-a}\,\kappa_{-b}\,\kappa'^{+a}\,\kappa'^{+b} + \bar{\kappa}_{+a}\,\bar{\kappa}_{+b}\,\bar{\kappa}'^{-a}\,\bar{\kappa}'^{-b}\big) - \frac{5}{2}\big(\kappa_{-a}\,\kappa_{-b}\,\bar{\kappa}'^{-a}\,\bar{\kappa}'^{-b} + \bar{\kappa}_{+a}\,\bar{\kappa}_{+b}\,\kappa'^{+a}\,\kappa'^{+b}\big)$$

$$-3\,\kappa_{-a}\,\bar{\kappa}'_{+b}\,\bar{\kappa}'^{-a}\,\kappa^{+b} - 4\big(\kappa_{-a}\,\bar{\kappa}_{+b}\,\kappa'^{+a}\,\bar{\kappa}'^{-b} - \kappa_{-a}\,\kappa'_{-b}\,\kappa'^{+a}\,\kappa^{+b} + \kappa_{-a}\,\bar{\kappa}'_{+b}\,\kappa^{+a}\,\bar{\kappa}'^{-b}$$

$$-\kappa_{-a}\,\kappa'_{-b}\,\bar{\kappa}^{-a}\,\bar{\kappa}'^{-b} - \bar{\kappa}_{+a}\,\bar{\kappa}'_{+b}\,\kappa^{+a}\,\kappa'^{+b} + \kappa_{-a}\bar{\kappa}'_{+b}\,\kappa'^{+a}\,\bar{\kappa}^{-b} + \bar{\kappa}_{+a}\,\kappa'_{-b}\,\bar{\kappa}'^{-a}\,\kappa^{+b}\big)$$

$$+5\big(\kappa_{-a}\,\kappa'_{-b}\,\bar{\kappa}'^{-a}\,\bar{\kappa}^{-b} + \bar{\kappa}_{+a}\,\bar{\kappa}'_{+b}\,\kappa'^{+a}\,\kappa^{+b}\big) + 6\,\kappa_{-a}\,\kappa'_{-b}\,\kappa^{+a}\,\kappa'^{+b} - 2\big(\kappa_{-a}\,\bar{\kappa}'_{+b}\,\bar{\kappa}^{-a}\,\kappa'^{+b}$$

$$+\bar{\kappa}_{+a}\,\kappa'_{-b}\,\kappa^{+a}\,\bar{\kappa}'^{-b}\big) - \frac{1}{2}(\bar{s}')^{\dot{a}}_a\,(\bar{s}')^{\dot{b}}_b\,s^a_{\dot{b}}\,s^b_{\dot{a}} + \frac{1}{2}s^a_{\dot{a}}\,s^b_{\dot{b}}\,(s')^{c,\dot{b}}\,s'^{d\,\dot{a}}\big(\epsilon_{ab}\,\epsilon_{cd} - \epsilon_{ad}\,\epsilon_{cb}\big)$$

$$-\frac{i}{4}\big(\kappa_{-a}\,\kappa'^{+a}\,\bar{s}^{\dot{a}}_b\,s^b_{\dot{a}} - \kappa_{-a}\,\kappa'^{+b}\,\bar{s}^{\dot{b}}_a\,s^a_{\dot{a}} - \kappa'_{-c}\,\kappa^{+a}\,\bar{s}^{\dot{a}}_a\,s^c_{\dot{a}}\big) - \frac{1}{2}\Big(\bar{\kappa}'_{+c}\,\kappa^{+c}\,\bar{s}^{\dot{a}}_a\,(s')^a_{\dot{a}} - \kappa'_{-c}\,\bar{\kappa}'^{-c}\,\bar{s}^{\dot{a}}_a\,s^a_{\dot{a}}$$

$$+\kappa'_{-c}\,\bar{\kappa}'^{-a}\,\bar{s}^{\dot{a}}_a\,s^c_{\dot{a}} + \bar{\kappa}'_{+c}\,\kappa'^{+a}\,\bar{s}^{\dot{a}}_a\,s^c_{\dot{a}} + \bar{\kappa}_{+a}\,\kappa^{+b}\,(\bar{s}')^{\dot{a}}_b\,(s')^a_{\dot{a}} - \bar{\kappa}'_{+c}\,\kappa^{+c}\,(\bar{s}')^{\dot{a}}_a\,s^a_{\dot{a}} - \kappa_{-a}\,\bar{\kappa}'^{-a}\,\bar{s}^{\dot{b}}_a\,(s')^b_{\dot{a}}$$

$$-\kappa_{-a}\,\bar{\kappa}^{-a}\,(\bar{s}')^{\dot{a}}_b\,(s')^b_{\dot{a}} + \kappa_{-a}\,\bar{\kappa}^{-b}\,(\bar{s}')^{\dot{a}}_b\,(s')^a_{\dot{a}}\Big) + \kappa_{-a}\,\bar{\kappa}'^{-a}\,(\bar{s}')^{\dot{a}}_b\,s^b_{\dot{a}} - \frac{3}{2}\Big(\kappa_{-a}\,\bar{\kappa}'^{-b}\,(\bar{s}')^{\dot{a}}_b\,s^a_{\dot{a}}$$

$$+\bar{\kappa}'_{+c}\,\kappa^{+a}\,(\bar{s}')^{\dot{a}}_a\,s^c_{\dot{a}}\Big) + \frac{1}{4}\epsilon^{bd}\,\epsilon_{\dot{a}b}\Big(i\big(\kappa_{-b}\,\bar{\kappa}^{-a}\,\bar{s}^{\dot{a}}_a\,(\bar{s}')^{\dot{b}}_d - \bar{\kappa}_{+d}\,\kappa^{+a}\,\bar{s}^{\dot{b}}_b\,(\bar{s}')^{\dot{a}}_a - 3\,\kappa_{-a}\,\bar{\kappa}^{-a}\,\bar{s}^{\dot{a}}_b\,(\bar{s}')^{\dot{b}}_d$$

$$+3\,\bar{\kappa}_{+a}\,\kappa^{+a}\,\bar{s}^{\dot{a}}_b\,(\bar{s}')^{\dot{b}}_d\big) + 2\big(\kappa_{-b}\,\kappa'^{+a}\,\bar{s}^{\dot{a}}_a\,(\bar{s}')^{\dot{b}}_d - \bar{\kappa}_{+d}\,\bar{\kappa}'^{-a}\,\bar{s}^{\dot{a}}_b\,(\bar{s}')^{\dot{b}}_a + \kappa'_{-d}\,\kappa'^{+a}\,\bar{s}^{\dot{a}}_a\,\bar{s}^{\dot{b}}_b$$

$$-\kappa_{-a}\,\kappa'^{+a}\,\bar{s}^{\dot{a}}_b\,(\bar{s}')^{\dot{b}}_d - \kappa_{-b}\,\kappa^{+a}\,(\bar{s}')^{\dot{a}}_a\,(\bar{s}')^{\dot{b}}_d + \kappa'_{-b}\,\kappa^{+a}\,\bar{s}^{\dot{a}}_a\,(\bar{s}')^{\dot{b}}_d\big) + 6\big(\bar{\kappa}_{+b}\,\bar{\kappa}'^{-a}\,\bar{s}^{\dot{a}}_a\,(\bar{s}')^{\dot{b}}_d$$

$$-\bar{\kappa}'_{+a}\,\bar{\kappa}^{-a}\,\bar{s}^{\dot{a}}_b\,(\bar{s}')^{\dot{b}}_d\big) - 8\,\bar{\kappa}'_{+d}\,\bar{\kappa}^{-a}\,\bar{s}^{\dot{a}}_a\,(\bar{s}')^{\dot{b}}_b\Big) + h.c.$$

Even though quite complicated, both $\mathcal{H}_{BB}$ and $\mathcal{H}_{FF}$ are definitely manageable expressions. Note that the pure bosonic Hamiltonian suffers from non derivative terms while the pure fermionic do not. For the latter, these were removed through the shift

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[20]The expression is not simplified by using the two spinor notation so we choose to present it with the s^a_b and $\kappa^{\pm}$ terms explicit.

(3.11). For the bosonic non derivative terms these can be removed through the use of a canonial transformation as explained in [48] and [26]. However, for the upcoming analysis, these will not have any effect on the calculations, so we choose to leave them as they stand.

As was explained in the previous section, the exact form of the fermionic shift relevant for the mixing Hamiltonian has not been determined. In the appendix we present the original Hamiltonian, prior to the fermionic shift, together with the form of $\widetilde{\Phi}$. The brave reader interested in the full mixing Hamiltonian can from there determine the exact form of the additional shift $\hat{\Phi}_3$. Having established the full shift one can, together with the corrections to the transverse part of π, determine the exact form of the shifted $\mathcal{H}_{BF}$.

We have now obtained the relevant Hamiltonian up to quartic order in number of fields. It is fully gauge fixed and posses the full SU(2|2)×U(1) symmetry of the theory. In the next two sections we will perform explicit calculations with it, starting by calculating the energy shift for a closed fermionic subsector and matching these with a set of light-cone Bethe equations.

4. Fermionic energy shifts and light-cone Bethe equations

In light of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence, energies of string excitations should correspond to anomalous dimensions of single trace operators in certain three dimensional Chern-Simons theories [2]. Based on integrability and the extensive knowledge from the original $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence [38], there has been a very rapid progress in understanding how to encode the spectral problem of both models in terms of Bethe equations. In [15] a all loop set of asymptotic Bethe equations were proposed for the full $\mathrm{OSP}(2,2|6)$ model which supposedly encode the energies of all possible (free) $\mathrm{AdS}_4 \times \mathbb{CP}_3$ string configurations. In [45] and [26] it was shown that the spectrum of string excitations in a closed bosonic subsectors of the theory exactly match the predictions of the Bethe equations from [15]. In this section we will extend this analysis to include fermionic operators. Not only will this be an important consistency check of the derived Hamiltonian, but it will also lend support to the assumed integrability of the full supersymmetric string model. It is also worth mentioning that this is the first explicit calculation probing the higher order fermionic sector of the duality.

Note that we will be rather brief in this section. For readers interested in the details, we refer to [52] and, especially, [26].

4.1 Strings in fermionic subsectors

In this section we will compute the energy shifts for a closed fermionic subsector constituted of the fields $\kappa^{\pm}$. Since we have cubic interaction terms in the Hamiltonian, the standard way to obtain the energy shifts would be through second order perturbation theory. However, this is quite an involved procedure since we have to

sum over intermediate zeroth order states. A much simpler approach is to remove the cubic terms through a unitary transformation of the Hamiltonian [48], [26]

$$\mathcal{H} \to e^{iV} \mathcal{H} e^{-iV}, \tag{4.1}$$

where the guiding principle for the construction of V is that it should obey

$$i[V, \mathcal{H}_2] = -\mathcal{H}_3, \tag{4.2}$$

and thus removes the unwanted terms.

To find an appropriate generating functional we need the oscillator components of $\mathcal{H}_3$

$$\sqrt{g}\,\mathcal{H}_3 = \mathcal{H}^{+++} + \mathcal{H}^{++-} + h.c \tag{4.3}$$
$$= \int dk\,dn\,dl \Big(C(k,n,l)^{+++}\, \bar{X}(k)\,\bar{Y}(n)\,\bar{Z}(l) + C(k,n,l)^{++-}\, \bar{X}(k)\,\bar{Y}(n)\,Z(l) \Big) + h.c.$$

where the oscillators X, Y and Z takes values in the set of $8_F + 8_B$ oscillators. However, since we want the energy shifts for $\kappa^\pm$ excitations, we only need the piece of $\mathcal{H}_3$ that depends quadratically on $\kappa^\pm$, that is, the first line of (3.18). Considering only this part, we can construct a function V with the property (4.2) as [48]

$$\sqrt{g}\,V = \int dk\,dn\,dl \Big\{ \tag{4.4}$$
$$\frac{-iC(k,n,l)^{+++}}{w_x(k) + w_y(n) + w_z(l)} \bar{X}(k)\,\bar{Y}(n)\,\bar{Z}(l) + \frac{-iC(k,n,l)^{++-}}{w_x(k) + w_y(n) - w_z(l)} \bar{X}(k)\,\bar{Y}(n)\,Z(l) \Big\} + h.c,$$

where $w_i(m)$ is either ω_m or Ω_m depending on the mass of $Z(l)$. It is straight forward, albeit tedious, to check that this choice of V indeed removes the cubic terms. However, from (4.1) it is clear the V commuted with the cubic part of the Hamiltonian will give rise to additional quartic terms,

$$\mathcal{H}_4^{Add} = -\frac{1}{2}\{V^2, \mathcal{H}_2\} + V\,\mathcal{H}_2\,V = \frac{i}{2}[V, \mathcal{H}_3]. \tag{4.5}$$

Even though the precise form of $\mathcal{H}_4^{Add}$ is quite complicated, evaluating its matrix elements is nevertheless significantly simpler than performing second order perturbation theory with the original Hamiltonian. Thus, after the unitary transformation, the Hamiltonian is of the form

$$\mathcal{H} = \mathcal{H}_2 + \frac{1}{g}\Big(\mathcal{H}_4 + \mathcal{H}_4^{Add}\Big) + \mathcal{O}(g^{-3/2}), \tag{4.6}$$

and this is the Hamiltonian we will use to calculate energy shifts in first order perturbation theory.

However, before we move on to that analysis there is one important issue we should comment on - namely, normal ordering. As was the case for the $\mathrm{AdS}_5 \times \mathrm{S}^5$

string, the next to leading order piece, which is the cubic contribution in our case, can be assumed to be normal ordered. The subleading piece can, however, not be assumed to be ordered. How to order them is an analysis that we have not performed since to the order of our interest, the normal ordering ambiguities can be addressed using ζ-function regularization, see [45] and [26][21]

The states we calculate the energy shifts from will be of the form

$$|m_1 \dots m_M \, n_1 \dots n_N\rangle = \bar{c}_1(m_1) \dots \bar{c}_1(m_M) \, \bar{d}^2(n_1) \dots \bar{d}^2(n_N) \, |0\rangle, \qquad (4.7)$$

where the sum of the mode numbers has to equal zero, $\sum_{i=1}^{M} m_i + \sum_{j=1}^{N} n_j = 0$. For simplicity we only consider states where all mode numbers are distinct.

The full quartic Hamiltonian, including the additional terms from the unitary transformation, have a general structure as

$$g\,\mathcal{H}_4 = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.8)$$
$$\frac{1}{(2\pi)^2} \int dk\,dn\,dl\,dm\, \delta(m+l-k-n)\Big\{ F(k,n,l,m)_{11}^{11}\, \bar{c}_1(k)\,\bar{c}_1(n)\,c^1(l)\,c^1(m)$$
$$+F(k,n,l,m)_{22}^{22}\, \bar{d}^2(k)\,\bar{d}^2(n)\,d_2(l)\,d_2(m) + F(k,n,l,m)_{21}^{12}\, \bar{c}_1(k)\,\bar{d}^2(n)\,d_2(l)\,c^1(m)\Big\}$$
$$+ \text{ Non relevant terms.}$$

The components $F(k,n,l,m)_{cd}^{ab}$ are quite complicated functions of the frequencies and the fermionic wave functions. Luckily, their form gets constrained considerably when projected on the states (4.7),

$$\Delta E = \langle n_N \dots n_1 \, m_M \dots m_1 |\, \mathcal{H}_4 \,|m_1 \dots m_M \, n_1 \dots n_N\rangle = \qquad\qquad (4.9)$$
$$\frac{1}{g}\Big\{ \frac{1}{16} \sum_{i,j=1}^{M} \frac{(m_i - m_j)^2}{\omega_{m_i}\,\omega_{m_j}} + \frac{1}{16} \sum_{i,j=1}^{N} \frac{(n_i - n_j)^2}{\omega_{n_i}\,\omega_{n_j}} + \frac{1}{8} \sum_{i=1}^{M} \sum_{j=1}^{N} \frac{(m_i + n_j)^2 + 4\,m_i\,n_j}{\omega_{m_i}\,\omega_{n_j}}\Big\}.$$

Since both the $\kappa^\pm$ part of (3.20) and the additional quartic terms are quite complicated, it is a remarkable feature of the uniform light-cone and kappa gauge that the energy shifts takes such a simple form.

In the next section we will show that these energy shifts are exactly reproduced from the asymptotic Bethe equations of [13] and [26].

4.2 Bethe equations

The starting point of this discussion will be the asymptotic light-cone Bethe equations

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[21]From the point of view of the worldsheet theory, calculating energy shifts to the order we are doing is basically a tree level calculation and the additional effects originating from the ordering terms enter at loop level.
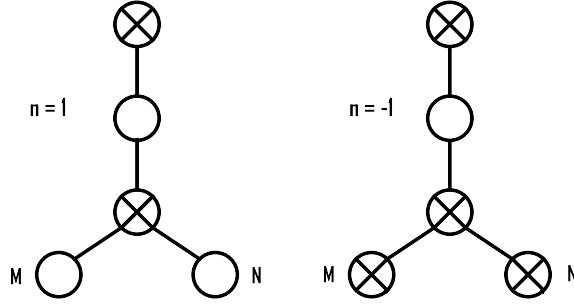
Figure 1: Dynkin diagrams for the two choices of gradings, $\eta = \pm 1$

of [26] given by[22]

$$\left(\frac{x^+(p_k)}{x^-(p_k)}\right)^{\frac{1}{2}\left(2g+\eta(M+N)\right)} = \tag{4.10}$$

$$\left(\frac{x^+(p_k)}{x^-(p_k)}\right)^{-g} \prod_{k\neq j}^{M} \left(\frac{x^+(p_k) - x^-(p_j)}{x^-(p_k) - x^+(p_j)}\right)^{\frac{1}{2}(1+\eta)} \sqrt{\frac{1 - \left(x^+(p_k)\,x^-(p_j)\right)^{-1}}{1 - \left(x^+(p_j)\,x^-(p_k)\right)^{-1}}} \times$$

$$\prod_{j=1}^{N} \left(\frac{x^+(p_k) - x^-(q_j)}{x^-(p_k) - x^+(q_j)}\right)^{\frac{1}{2}(1-\eta)} \sqrt{\frac{1 - \left(x^+(q_j)\,x^-(p_k)\right)^{-1}}{1 - \left(x^+(p_k)\,x^-(q_j)\right)^{-1}}} + \mathcal{O}(g^{-3}),$$

$$\left(\frac{x^+(q_k)}{x^-(q_k)}\right)^{\frac{1}{2}\left(2g+\eta(M+N)\right)} =$$

$$\left(\frac{x^+(q_k)}{x^-(q_k)}\right)^{-g} \prod_{k\neq j}^{M} \left(\frac{x^+(q_k) - x^-(q_j)}{x^-(q_k) - x^+(q_j)}\right)^{\frac{1}{2}(1+\eta)} \sqrt{\frac{1 - \left(x^+(q_k)\,x^-(q_j)\right)^{-1}}{1 - \left(x^+(q_j)\,x^-(q_k)\right)^{-1}}} \times$$

$$\prod_{j=1}^{N} \left(\frac{x^+(q_k) - x^-(p_j)}{x^-(q_k) - x^+(p_j)}\right)^{\frac{1}{2}(1-\eta)} \sqrt{\frac{1 - \left(x^+(p_j)\,x^-(q_k)\right)^{-1}}{1 - \left(x^+(q_k)\,x^-(p_j)\right)^{-1}}} + \mathcal{O}(g^{-3}).$$

The constant $\eta = \pm 1$ selects one of the two Dynkin diagrams in figure 1. In [26] only the Bethe equations for the $\eta = 1$ diagram were spelled out but it should be clear from [40] and, especially, [52] how the generalization from the bosonic case, in [26], to the situation at hand works. The main difference between the two diagrams is the statistics of the M and N nodes, where the integers denote the number of $\bar{c}_1$ and $\bar{d}^2$ excitations. For $\eta = 1$ the basic spin flips in the two spin chains are purely bosonic while for $\eta = -1$ they are fermionic. Since we are calculating energy shifts for fermionic operators, we need to choose $\eta = -1$[23].

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[22]In [26] a large light-cone momentum, $P_+ \to \infty$ and $\widetilde{\lambda}' \sim$ constant, expansion was utilized. With the identifications $P_+ = 2g$ and $\widetilde{\lambda}' = 1$, that expansion is equivalent to the strong coupling expansion used in this paper.

[23]If not, then the outer arms of the Bethe equations gets excited and the functional form of the equations are considerably more complicated. For an example, see [52].

The spectral parameters $x^\pm(p_k)$ in (4.10), can be solved for through

$$x^\pm(p_k) + \frac{1}{x^\pm(p_k)} = \frac{1}{h(\lambda)}\left(\phi(p_k) \pm \frac{i}{2}\right), \tag{4.11}$$

where

$$\phi(p_k) = \cot\frac{p_k}{2}\sqrt{\frac{1}{4} + 4\,h(\lambda)^2\,\sin^2\frac{p_k}{2}}. \tag{4.12}$$

The function $h(\lambda)$ is a novel feature for the AdS$_4$ / CFT$_3$ duality and is, so far, only known perturbatively[24] [4], [8], [43]. It scales differently in the weak / strong coupling regimes, where in our case we only need the leading order part of the strong coupling expansion

$$h(\lambda) = \sqrt{\frac{\lambda}{2}} + \mathcal{O}(\lambda^0), \tag{4.13}$$

where the 't Hooft coupling λ is related to g as

$$\lambda = \frac{g^2}{2\pi^2}. \tag{4.14}$$

The two spin chains in (4.10) are related through a momentum, or cyclicity, constraint

$$1 = \prod_{k=1}^{M}\frac{x^+(p_j)}{x^-(p_j)}\prod_{j=1}^{N}\frac{x^+(q_j)}{x^-(q_j)}. \tag{4.15}$$

The light-cone energy, corresponding to eigenvalues of $-p_-$ on the string theory side, can be expressed through

$$E = \Delta - J = \tag{4.16}$$
$$\sum_{j=1}^{M}\left(\sqrt{\frac{1}{4} + 4\,h(\lambda)^2\sin^2\frac{p_j}{2}} - \frac{1}{2}\right) + \sum_{j=1}^{N}\left(\sqrt{\frac{1}{4} + 4\,h(\lambda)^2\sin^2\frac{q_j}{2}} - \frac{1}{2}\right).$$

It is the functional form of the light-cone energy that we will match against the string energy shifts in (4.9). To achieve this we assume a perturbative expansion for the momentas,

$$p_k = \frac{p_k^0}{2g} + \frac{p_k^1}{(2g)^2}, \qquad q_k = \frac{q_k^0}{2g} + \frac{q_k^1}{(2g)^2}, \tag{4.17}$$

which allows us to perturbatively solve for p_k and q_k in (4.10). For the leading order contribution one finds

$$p_k^0 = 4\pi\,m_k, \qquad q_k^0 = 4\,\pi\,n_k. \tag{4.18}$$

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[24]The reason we could ignore the ordering issues of the light-cone Hamiltonian, is because they kick in at order $\mathcal{O}(\lambda^0)$ of $h(\lambda)$, i.e. beyond the tree level approximation.

The higher order components p_k^1 and q_k^1 are a bit more involved but can straightforwardly be deduced from (4.10).

Using the solutions in (4.16) and expanding gives that the $\mathcal{O}(g^{-1})$ shifts are given by[25]

$$\Delta E = \tag{4.19}$$

$$\frac{1}{4g}\sum_{k=1}^{M}\left\{\frac{(M+N)m_k^2}{\omega_k} + \frac{8\,m_k^2}{\omega_k}\left(\sum_{j=1}^{M}\frac{m_j(m_k - m_j)}{(1+2\omega_k)(1+2\omega_j) - 4\,m_k\,m_j}\right.\right.$$

$$\left.\left. -\sum_{j=1}^{N}\frac{n_j(m_k - n_j)}{(1+2\omega_k)(1+2\omega_j) - 4\,m_k\,n_j} - \sum_{j=1}^{N}\frac{n_j(1+\omega_j + \omega_k)}{n_j(1+2\omega_k) - m_k(1+2\omega_j)}\right)\right\}$$

$$+\frac{1}{4g}\sum_{k=1}^{N}\left\{\frac{(M+N)n_k^2}{\omega_k} + \frac{8\,n_k^2}{\omega_k}\left(\sum_{j=1}^{N}\frac{n_j(n_k - n_j)}{(1+2\omega_k)(1+2\omega_j) - 4\,n_k\,n_j}\right.\right.$$

$$\left.\left. -\sum_{j=1}^{M}\frac{m_j(n_k - m_j)}{(1+2\omega_k)(1+2\omega_j) - 4\,n_k\,m_j} - \sum_{j=1}^{M}\frac{m_j(1+\omega_j + \omega_k)}{m_j(1+2\omega_k) - n_k(1+2\omega_j)}\right)\right\},$$

which should be augmented with the expanded cyclicity condition (4.15),

$$\sum_{j=1}^{M} m_j + \sum_{j=1}^{N} n_j = 0. \tag{4.20}$$

After enforcing this constraint in (4.19) one can show that the energy shifts calculated from the Bethe equations (4.10) precisely matches the string energies obtained from diagonalizing the string Hamiltonian in (4.9). However, as also was the case for the bosonic calculation in [26], it is quite tedious to show the algebraic equivalence of the two expressions. The use of a computer program able to handle symbolic manipulations is recommended.

As a further consistency check of the Hamiltonian we derived, we have also verified the result of [26] with the $\eta = 1$ grading using the purely bosonic cubic and quartic Hamiltonian (3.18) and (3.19).

This is the first calculation beyond leading order probing the fermionic sector of the $\text{AdS}_4 \times \mathbb{CP}_3$ string. Not only does it establish the validity of the derived Hamiltonian but it also lend strong support to the assumed integrability, and especially, the factorization properties following from that.

Before we end this section let us also mention the result in [35]. There the authors constructed a fermionic reduction to a subsector identical to the SU(1|1) sector of the $\text{AdS}_5 \times \text{S}^5$ string [47] [46]. However, this is not the sector we have studied since the form of the Bethe equations are not the same as the SU(1|1) $\subset$ PSU(2, 2|4) light-cone Bethe equations in [48]. The relation between the two sectors is unclear for us and it would be nice to understand it further.

[25]We abbreviated $\omega_{m_k} = \omega_k$ and similar for the n_k indices. Which excitation the index belong to should be clear from the context.

1a

1b

Figure 2: Self energy graphs.

5. Quantum corrections to the heavy modes

The Bethe equations presented in the earlier section can be extended to the full symmetry group OSP(2,2|6) in which the Bethe roots fall into short representations of SU(2|2), that is, only $4_F + 4_B$ modes appear as fundamental excitation in the scattering matrix. At leading order these have the magnon dispersion relation, $\omega = \sqrt{\frac{1}{4} + p^2}$, so it is natural to associate these with the $4_F + 4_B$ light string modes, $\kappa^\pm$ and $\omega_{\dot{a}}$. However, as we have seen, critical string theory exhibits $8_F + 8_B$ oscillatory degrees of freedom, so how are we to understand the modes y, z_i and s_b^a? From the quadratic Lagrangian it certainly seems like they are on an equal footing as the light modes, so why do they not appear as excitations in the S-matrix?

By continuing a line of research initiated by Zarembo in [11] we will try to address this question in the upcoming section. We will do this by calculating loop corrections to the propagators of the massive fields. As we will argue, the loop corrections have the effect that the pole gets shifted beyond the energy threshold for pair production of two light modes, so the heavy state dissolves into a two particle continuum.

From the analysis in the previous section, it is clear that the two type of relevant loop diagrams are a three vertex loop from (3.18) and a tadpole diagram from the full quartic Hamiltonian, see Figure 2. To calculate the corrections one would need to calculate the full contribution from both types of diagram. However, for the question wetter the heavy modes come as fundamental excitations or not, it is enough to focus our attention on the propagators analytic properties close to the pole. For the pure quadratic theory, at strictly infinite coupling, the massive propagators has a pole at $\bar{k}^2 = 1$. Incorporating quantum corrections, it gets shifted as

$$\Delta(k) \sim \int d^2k \frac{Z(k)}{\bar{k}^2 - 1 + \frac{1}{g}\delta m + i\epsilon},$$

where, as we will show, the relevant part of the mass corrections are of the form

$$\delta m = C(k)\sqrt{1 - \frac{1}{\bar{k}^2}}. \tag{5.1}$$

For values of $\bar{k}$ such that the difference $\bar{k}^2 - 1$ is very small, the first term in the propagator can be as important as the second one. Since the bare pole lies exactly

at the branch point for pair production of two light modes, the sign of $C(k)$ may change the analytical properties of $\Delta(k)$. If the sign is positive, then the one particle pole is shifted below the threshold energy. If negative, however, the pole gets shifted beyond the threshold energy and disappears. This means that this field does not exist as a physical excitation for finite values of the coupling g.

As is well known, the behavior of a Feynman integral close to its pole is dominated by its imaginary part. Thus, the behavior of the quantum corrected pole can be extracted from the imaginary part of δm. This has the pleasant advantage that, for the calculation at hand, we can neglect the tadpole diagrams. This is easy to understand if one takes a look at the general structure of such a contribution,

$$\int d^2k \frac{G(k)}{\bar{k}^2 - m^2 + i\,\epsilon},$$

where $g(k)$ is a even polynomial in k and m is the mass of the particle in the loop. By direct inspection it is clear that there are no extra branch points associated to this integral. Of course, there are however a lot of real terms, both finite and divergent, resulting from the integral. It is however likely that supersymmetry guarantees that these terms cancel among themselves[26].

The analysis then boils down to isolating the imaginary part of the three vertex loops. Since we will only focus on the massive bosonic coordinates, the relevant part of the cubic Hamiltonian (3.18) is

$$\sqrt{g}\,\mathcal{H}_3^{loop} = \left(\bar{\Psi}_a\,\Psi^b\right)'Z_b^a + i\left(\bar{\Psi}\,\gamma^1\,\Psi' - \bar{\Psi}'\,\gamma^1\,\Psi\right)_a^b(Z')_b^a - 2i\left(\bar{\Psi}'\,\Psi - \bar{\Psi}\,\Psi'\right)_a^b P_{z,b}^a, \quad (5.2)$$

from where its clear that the the fields in the loops are $\omega_{\dot{a}}$ for the singlet and $\kappa^{\pm}$ for Z_b^a.

For the upcoming analysis we will need the bare propagators,

$$\langle 0|T\{\Psi(\bar{\sigma})\,\bar{\Psi}(\bar{\sigma}')\}|0\rangle = \frac{i}{2(2\pi)^2}\int d^2p \frac{\left(\gamma^{\alpha}\,p_{\alpha} + \frac{1}{2}\right)}{\bar{p}^2 - \frac{1}{4} + i\epsilon}e^{-i\bar{p}\cdot\hat{\sigma}}, \qquad (5.3)$$

$$\langle 0|T\{\chi(\bar{\sigma})\,\bar{\chi}(\bar{\sigma}')\}|0\rangle = \frac{i}{2(2\pi)^2}\int d^2p \frac{\left(\gamma^{\alpha}\,p_{\alpha} + 1\right)}{\bar{p}^2 - 1 + i\epsilon}e^{-i\bar{p}\cdot\hat{\sigma}},$$

$$\langle 0|T\{Z_b^a(\bar{\sigma})\,Z_d^c(\bar{\sigma}')\}|0\rangle = \frac{i}{(2\pi)^2}\int d^2p \frac{\left(2\,\delta_d^a\,\delta_b^c - \delta_b^a\,\delta_d^c\right)}{\bar{p}^2 - 1 + i\epsilon}e^{-i\bar{p}\cdot\hat{\sigma}},$$

$$\langle 0|T\{y(\bar{\sigma})\,y(\bar{\sigma}')\}|0\rangle = \frac{i}{(2\pi)^2}\int d^2p \frac{e^{-i\bar{p}\cdot\hat{\sigma}}}{\bar{p}^2 - 1 + i\epsilon},$$

$$\langle 0|T\{\omega_{\dot{a}}(\bar{\sigma})\,\bar{\omega}^{\dot{b}}(\bar{\sigma}')\}|0\rangle = \frac{2\,i}{(2\pi)^2}\int d^2p \frac{\delta_{\dot{a}}^{\dot{b}}}{\bar{p}^2 - \frac{1}{4} + i\epsilon}e^{-i\bar{p}\cdot\hat{\sigma}},$$

where we abbreviated $\hat{\sigma} = \bar{\sigma} - \bar{\sigma}'$ and $\bar{\sigma} = (\tau, \sigma)$.

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[26]The heavy modes are in a semi short representation of the SU(2|2) and should be BPS protected from mass renormalizations [11] [42].

5.1 Massive singlet

We will start the analysis with the massive singlet y, already calculated by Zarembo in [11]. The analysis basically boils down to determining the sign of the mass correction and since we will encounter (complex) multi valued functions, some care is asked for when determining which value to take as physical. For this reason we will be rather detailed in this part of the calculation.

For the singlet we find that one loop corrected propagator equals

$$\langle \Omega | T\big(y(x)\, y(y)\big) | \Omega \rangle = \frac{i}{(2\pi)^2} \int d^2k \frac{e^{-i\bar{k}\cdot(\bar{x}-\bar{y})}}{\bar{k}^2 - 1 + i\epsilon}\Big(1 - \frac{1}{\bar{k}^2 - 1 - i\epsilon}\pi_{00}\Big), \quad (5.4)$$

where the polarization tensor is given by

$$-\pi_{00} = \frac{i}{2(2\pi)^2} \int d^2p \frac{\big(2p_0 - k_0\big)^2}{\big(\bar{p}^2 - \frac{1}{4} + i\epsilon\big)\big((\bar{p} - \bar{k})^2 - \frac{1}{4} + i\epsilon\big)} \quad (5.5)$$

Using the standard Feynman parameterization with $\bar{q} = \bar{p} - \bar{k}\,z$, a direct computation gives

$$-\pi_{00} = \quad (5.6)$$
$$\frac{1}{4\pi}\Big\{\frac{1}{\eta} - \gamma - \log(\pi) - \int_0^1 dz\Big(\log\big(\frac{1}{4} - \bar{k}^2(1 - z)z + i\epsilon\big) + \frac{(1 - 2z)^2 k_0^2}{2(\frac{1}{4} - \bar{k}^2(1 - z)z + i\epsilon)}\Big)\Big\},$$

where we used dimensional regularization to isolate the divergence. For a purely real argument the logarithm develops a imaginary $\pm i\pi$ part when $\bar{k}^2 > 1$, and to isolate it, we integrate z over the interval $\frac{1}{2}(1 \pm \sqrt{1 - \frac{1}{\bar{k}^2}})$. With the ϵ prescription included, we find that it gives rise to a small positive imaginary contribution, so it is the $i\pi$ part of $\text{Im}\,(\log)$ that we should use. Thus, for $\bar{k}^2 > 1$, its imaginary contribution is

$$\text{Im}\Big[\frac{1}{4\pi} \int \log(\frac{1}{4} - \bar{k}^2(1 - z)z)\Big] = \frac{1}{4}\sqrt{1 - \frac{1}{\bar{k}^2}}. \quad (5.7)$$

If we introduce the short hand notation $\alpha = \frac{1}{2}\sqrt{1 - \frac{1}{\bar{k}^2} - \frac{4\,i\epsilon}{\bar{k}^2}}$ and shift $z \to y + \frac{1}{2}$, the last term in (5.6) can be written as

$$\frac{k_0^2}{4\pi\,\bar{k}^2} \int_0^{\frac{1}{2}} dy\Big(4 + \frac{2\alpha}{y - \alpha} - \frac{2\alpha}{y + \alpha}\Big). \quad (5.8)$$

The imaginary part of this integral comes from the middle term, where the ϵ prescription gives a negative imaginary contribution. To calculate the imaginary part of (5.8) we introduce $y - \alpha = \epsilon_0\, e^{i\theta}$, which gives

$$-\frac{k_0^2}{4\,\bar{k}^2}\sqrt{1 - \frac{1}{\bar{k}^2}} = -\frac{k_0^2}{4}\sqrt{1 - \frac{1}{\bar{k}^2}} + \mathcal{O}(1 - \frac{1}{\bar{k}^2})^{\frac{3}{2}}, \quad (5.9)$$

where we assumed that $\bar{k}^2$ is close to the two particle threshold.

Combining the two results shows that

$$\operatorname{Im}\pi_{00} = \frac{1}{4}(1 - k_0^2)\sqrt{1 - \frac{1}{\bar{k}^2}} = -\frac{1}{4}k_1^2\sqrt{1 - \frac{1}{\bar{k}^2}} + \mathcal{O}(1 - \frac{1}{\bar{k}^2})^{\frac{3}{2}}, \qquad (5.10)$$

which is negative definite close to the pole. This is almost what Zarembo calculated in [11]. The difference lies in the form of the square root, which in [11] was, $\sqrt{1 - \bar{k}^2}$, while we have $\sqrt{1 - \frac{1}{\bar{k}^2}}$. This is related to the expansion scheme and has no physical consequence. What is important is the presence of a positive definite function with the correct overall sign in front.

5.2 Massive AdS coordinates

Having established what happens to the singlet when loop corrections are taken into account we turn next to the remaining massive coordinates. The corrected propagator we want to calculate is

$$\langle\Omega|T\big(Z_l^k(x)\, Z_n^m(y)\big)|\Omega\rangle. \qquad (5.11)$$

For this calculation, it is convenient to write the relevant part of the cubic Hamiltonian, (3.18), as

$$\sqrt{g}\,\mathcal{H}_3 = i\big(\bar{\Psi}\,\gamma^1\,\Psi' - \bar{\Psi}'\gamma^1\,\Psi + i\,\bar{\Psi}\cdot\Psi\big)_a^b(Z')_b^a - 2i\big(\bar{\Psi}'\,\Psi - \bar{\Psi}\,\Psi'\big)_a^b(P_z)_b^a. \qquad (5.12)$$

Due to the fermions in the loop, we will encounter quadratic divergences along the way. However, as was the case for the singlet, these will not contribute to the imaginary part.

Due to the more complicated cubic Hamiltonian, the calculation will be more involved. However, pushing through with the calculation and using the Feynman parameterization as before, gives that the relevant terms are of the form

$$\delta m = \int d^2q \frac{F_0(\bar{k}) + F_2(\bar{k}, q_0^2, q_1^2) + F_4(\bar{k}, q_0^2\, q_1^2, q_1^4)}{(\bar{q}^2 - \bar{k}^2(1 - z)z - \frac{1}{4} - i\,\epsilon)^2}, \qquad (5.13)$$

where the subscript denote the power of q_i in the nominator.

To determine the form of the functions F_i, we repeat the same procedure as for the singlet computation. Unfortunately they are rather involved so we will not present them explicitly, but a straight forward, albeit somewhat tedious, calculation shows that

$$\delta m_0 = -2\,k_1^2\sqrt{1 - \frac{1}{\bar{k}^2}}, \qquad \delta m_2 = \frac{1}{3}\big(k_0^2 - k_0^4 + 4k_1^2 + k_1^4\big)\sqrt{1 - \frac{1}{\bar{k}^2}}, \qquad (5.14)$$

$$\delta m_4 = \frac{1}{3}(\bar{k}^2 - 1)(k_0^2 + k_1^2)\sqrt{1 - \frac{1}{\bar{k}^2}},$$

which added together gives

$$\delta m = -k_1^2 \sqrt{1 - \frac{1}{\bar{k}^2}} \left(2\delta_n^k \, \delta_l^m - \delta_l^k \, \delta_n^m \right), \tag{5.15}$$

which is strictly negative[27] and exact for $\bar{k}^2 > 1$.

With this we conclude that all the massive bosons dissolve in a two particle continuum.

5.3 Massive fermions and comments

Even though we have not performed the calculation in detail, it is plausible that the massive s_a^b fields exhibit the same property as the massive bosons. By direct inspection of the cubic Hamiltonian it is clear that the fields in the loop will be the two light $\omega_{\dot{a}}$ and $\kappa^\pm$. Unfortunately, due to the rather entangled mixing between the s_a^b and $\kappa^\pm$ fields, the imaginary part of the propagator is rather involved. Nevertheless, it is still of the form

$$C(k) \sqrt{1 - \frac{1}{\bar{k}^2}} + \mathcal{O}(1 - \frac{1}{\bar{k}^2})^{\frac{3}{2}},$$

with a complicated $C(k)$ which we have not determined. Instead of pursuing this line of research, a much better way to approach the problem would be to calculate the worldsheet scattering matrix and from there study the behavior of the massive fields. Unfortunately, since it is only through loop corrections that the physical role of the massive fields emerge, the calculation of the scattering matrix would be complicated. In fact, not even for the $\text{AdS}_5 \times \text{S}^5$ case is the one loop BMN scattering matrix fully known. This gives a rather grim outlook for the possibility of deriving the exact one loop behavior of the $\text{AdS}_4 \times \mathbb{CP}_3$ BMN string.

6. Summary and closing comments

In this paper we have presented a detailed discussion about the type IIA superstring in $\text{AdS}_4 \times \mathbb{CP}_3$. By starting directly from the $\mathfrak{osp}(2,2|6)$ superalgebra we constructed the string Lagrangian through its graded components. Then by identifying the bosonic subalgebra commuting with the light-cone Hamiltonian, we introduced a notation covariant under the gauge fixed $\text{SU}(2|2) \times \text{U}(1)$ symmetry. The covariant string Lagrangian was the starting point for a perturbative analysis in a strong coupling limit where we almost immediately ran into problem due to the presence of higher order kinetic terms for the fermionis. These had the sad effect that they complicated the general structure of the theory to such an extent that we only presented parts of the canonical Hamiltonian. Nevertheless, we proceeded with a calculation of

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[27]Or, to be precise, it is strictly negative when we restrict to the z_i propagation, $< T(z_i \, z_j) >$.

energy shifts for fermionic string configurations built out of a arbitrary number of c_1 and d^2 oscillators. These shifts we successfully matched with the prediction coming from a conjectured set of light-cone Bethe equations.

We then moved on to an investigation of the role of the massive bosonic modes. By calculating loop corrections to the propagators of the massive fields we saw that the massive modes dissolved into a two particle continuum.

We also provided an extensive appendix where the original Hamiltonian, including the kinetic terms of the fermions were spelled out in detail.

All in all we have presented a rather thorough study of the $AdS_4 \times \mathbb{CP}_3$ superstring. Naturally a lot remains to be done, where perhaps the most stressing, at least from the point of view of our analysis, is to establish the one loop scattering matrix for the heavy modes. Even though we provided arguments for that the heavy modes dissolve in a two particle continuum, it would be desirable to see it explicitly in terms of Feynman diagrams. Unfortunately, due to the complexity of the theory, it does not seem very plausible that one can achieve this through the use of the BMN string. Perhaps a better way to approach the problem would be through the so called near flat space limit [54], [55].

Another interesting line of research would be to consider higher order corrections to the interpolating function $h(\lambda)$ that occurs in the magnon dispersion relation. It has been extensively studied in [16] [17] [18] [19], but its higher order structure remains unknown.

We plan to return to some of these questions in further investigations.

Acknowledgments

A. Matrices

In this appendix we present the exact form of some of the matrices encountered in the main text. The conventions are those of Arutyunov and Frolov in [9].

Starting out with the SO(1,3) Γ we have

$$\Gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \Gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \Gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \Gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

satisfying $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$ with signature (+,-,-,-). As usual, the combinations $\Gamma^{\mu\nu} = \frac{1}{2}[\Gamma^\mu, \Gamma^\nu]$ generate the lie algebra $\mathfrak{so}(1,3)$.

The charge conjugation matrix, C_4, can be expressed in terms of the Γ matrices as

$$C_4 = i\,\Gamma^0\,\Gamma^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

The six T_i matrices are generators of $\mathfrak{so}(6)$ along $\mathbb{CP}_3$ and are given by

$$T_1 = E_{13} - E_{31} - E_{24} + E_{42}, \qquad T_2 = E_{14} - E_{41} + E_{23} - E_{32}, \qquad (A.1)$$
$$T_3 = E_{15} - E_{51} - E_{26} + E_{62}, \qquad T_2 = E_{16} - E_{61} + E_{25} - E_{52},$$
$$T_5 = E_{35} - E_{53} - E_{46} + E_{64}, \qquad T_2 = E_{36} - E_{63} + E_{45} - E_{54},$$

where E_{ij} is the 6×6 matrix with all elements zero except the i, j'th component which is unity. The normalization is as follows,

$$Tr(T_i\,T_j) = -4\,\delta_{ij}. \qquad (A.2)$$

The T_i matrices satisfy the following important properties,

$$\{T_1, T_2\} = 0, \qquad \{T_3, T_4\} = 0, \qquad \{T_5, T_6\} = 0. \qquad (A.3)$$

In the text we frequently make use of the complex combinations,

$$\tau_1 = \frac{1}{2}\big(T_1 - i\,T_2\big), \qquad \tau_2 = \frac{1}{2}\big(T_3 - i\,T_4\big), \qquad (A.4)$$

and $\bar{\tau}_i$ for conjugated combinations.

B. Mixing term of the original Hamiltonian

In this appendix we present the full, non shifted, quartic Hamiltonian, which combined with the fermionic kinetic term in (3.9) encodes the full dynamics of the quartic theory.

We start out by presenting the original cubic Hamiltonian which is similar but not identical to the shifted one,

$$\sqrt{g}\,\mathcal{H}_3^{ns} = \tag{B.1}$$
$$\left(\bar{\Psi}'\cdot\Psi\right)_b^a Z_a^b + \left(\bar{\Psi}\,\gamma^1\,\Psi'\right)_b^a (Z')_a^b + i\,y\,\omega_{\dot{a}}\,\bar{p}^{\dot{a}} + 3i\left(\kappa_{-\,a}\,\bar{s}^{a\,\dot{a}} - \bar{\kappa}_{+\,a}\,s^{a\,\dot{a}}\right)p_{\dot{a}}$$
$$+\left(\frac{3}{8}(\bar{\kappa}_{+\,a}\,s^{a\,\dot{a}} + \kappa_{-\,a}\,\bar{s}^{a\,\dot{a}}) + i(\kappa'_{-\,a}\,s^{a\,\dot{a}} - \bar{\kappa}'_{+\,a}\,\bar{s}^{a\,\dot{a}}) + \frac{i}{2}(\bar{\kappa}_{+\,a}\,(\bar{s}')^{a\,\dot{a}} - \kappa_{-\,a}\,(s')^{a\,\dot{a}})\right)\omega_{\dot{a}}$$
$$+\frac{1}{2}\left(\kappa_{-\,a}\,(\bar{s}')^{a\,\dot{a}} - \kappa'_{-\,a}\,\bar{s}^{a\,\dot{a}} - \bar{\kappa}'_{+\,a}\,s^{a\,\dot{a}} + \bar{\kappa}_{+\,a}\,(s')^{a\,\dot{a}}\right)\omega'_{\dot{a}} + h.c,$$

where the *ns* superscript denotes that this is the non shifted Hamiltonian.

Next we turn to the quartic interactions, where we as before split up the Hamiltonian according to its field content. The pure bosonic part will naturally be identical to (3.19) so we will not present it again. For the pure fermionic part we find

$$g\,\mathcal{H}_{FF}^{ns} = \kappa_{-\,a}\,\bar{\kappa}_{+\,b}\,\kappa^{+\,b}\,\bar{\kappa}^{-\,a} - \kappa_{-\,a}\,\bar{\kappa}_{+\,b}\,\kappa'^{+\,a}\,\bar{\kappa}'^{-\,b} - \kappa_{-\,a}\,\kappa'_{-\,b}\,\kappa^{+\,b}\,\kappa'^{+\,a} \tag{B.2}$$
$$-\kappa_{-\,a}\,\kappa'_{-\,b}\,\bar{\kappa}^{-\,b}\,\bar{\kappa}'^{-\,a} - \kappa_{-\,a}\,\bar{\kappa}'_{+\,b}\,\kappa^{+\,b}\,\bar{\kappa}'^{-\,a} - \kappa_{-\,a}\,\bar{\kappa}'_{+\,b}\,\bar{\kappa}^{-\,b}\,\kappa'^{+\,a} - \bar{\kappa}_{+\,a}\,\kappa'_{-\,b}\,\kappa^{+\,b}\,\bar{\kappa}'^{-\,a}$$
$$-\bar{\kappa}_{+\,a}\,\bar{\kappa}'_{+\,b}\,\kappa^{+\,b}\,\kappa'^{+\,a} + \frac{1}{2}\Big(\kappa_{-\,a}\,\kappa_{-\,b}\,\bar{\kappa}^{-\,a}\,\bar{\kappa}^{-\,b} - \kappa_{-\,a}\,\kappa_{-\,b}\,\bar{\kappa}'^{-\,a}\,\bar{\kappa}'^{-\,b} + \bar{\kappa}_{+\,a}\,\bar{\kappa}_{+\,b}\,\kappa^{+\,a}\,\kappa^{+\,b}$$
$$-\bar{\kappa}_{+\,a}\,\bar{\kappa}_{+\,b}\,\kappa'^{+\,a}\,\kappa'^{+\,b}\Big) - \frac{3}{2}\Big(\kappa_{-\,a}\,\kappa_{-\,b}\,\kappa'^{+\,a}\,\kappa'^{+\,b} + \bar{\kappa}_{+\,a}\,\bar{\kappa}_{+\,b}\,\bar{\kappa}'^{-\,a}\,\bar{\kappa}'^{-\,b}\Big) + 2\Big(\kappa_{-\,a}\,\kappa'_{-\,b}\,\kappa^{+\,a}\,\kappa'^{+\,b}$$
$$-\kappa_{-\,a}\,\bar{\kappa}_{+\,b}\,\kappa^{+\,a}\,\bar{\kappa}^{-\,b} + \kappa_{-\,a}\,\bar{\kappa}'_{+\,b}\,\bar{\kappa}^{-\,a}\,\kappa'^{+\,b} + \bar{\kappa}_{+\,a}\,\kappa'_{-\,b}\,\kappa^{+\,a}\,\bar{\kappa}'^{-\,b}\Big) + 3\,\kappa_{-\,a}\,\bar{\kappa}_{+\,b}\,\kappa'^{+\,b}\,\bar{\kappa}'^{-\,a}$$
$$+i\Big(\kappa_{-\,a}\,\bar{\kappa}_{+\,b}\,\bar{\kappa}^{-\,b}\,\bar{\kappa}'^{-\,a} + \kappa_{-\,a}\,\bar{\kappa}_{+\,b}\,\kappa^{+\,a}\,\kappa'^{+\,b} + \frac{1}{2}\Big(\kappa_{-\,a}\,\bar{\kappa}_{+\,b}\,\kappa^{+\,b}\,\kappa'^{+\,a} + \kappa_{-\,a}\,\bar{\kappa}_{+\,b}\,\bar{\kappa}^{-\,a}\,\bar{\kappa}'^{-\,b}\Big)$$
$$-3i\Big(\kappa_{-\,a}\,\kappa_{-\,b}\,\kappa^{+\,a}\,\bar{\kappa}'^{-\,b} + \kappa_{-\,a}\,\bar{\kappa}'_{+\,b}\,\kappa^{+\,a}\,\kappa^{+\,b}\Big) + i\frac{7}{2}\Big(\bar{\kappa}_{+\,a}\,\bar{\kappa}_{+\,b}\,\kappa^{+\,a}\,\bar{\kappa}'^{-\,b} - \kappa_{-\,a}\,\kappa_{-\,b}\,\bar{\kappa}^{-\,a}\,\kappa'^{+\,b}\Big)\Big)$$
$$+\frac{i}{2}\epsilon^{\dot{a}\dot{b}}\,\epsilon_{ab}\Big(s_{\dot{a}}^a\,s_{\dot{b}}^c\,s_{\dot{d}}^b\,(\bar{s}')_c^{\dot{d}} + \frac{1}{2}(s_{\dot{a}}^c\,s_{\dot{b}}^a\,(s')_{\dot{d}}^b\,\bar{s}_c^{\dot{d}} - s_{\dot{d}}^a\,s_{\dot{a}}^b\,(s')_{\dot{b}}^c\,\bar{s}_c^{\dot{d}})\Big) + \epsilon^{\dot{a}\dot{b}}\,\epsilon_{bc}\Big(-i\,s_{\dot{a}}^a\,s_{\dot{b}}^b(\kappa_{-\,a}\,\bar{\kappa}'^{-\,c}$$
$$+\bar{\kappa}'_{+\,a}\,\kappa^{+\,c} - \frac{i}{2}\,\bar{\kappa}'_{+\,a}\,\bar{\kappa}'^{-\,c}) + \frac{1}{2}s_{\dot{a}}^b\,(s')_{\dot{b}}^a(\bar{\kappa}_{+\,a}\,\bar{\kappa}'^{-\,c} + \bar{\kappa}'_{+\,a}\,\bar{\kappa}^{-\,c} - i\,\kappa_{-\,a}\,\bar{\kappa}^{-\,c} - i\,\bar{\kappa}_{+\,a}\,\kappa^{+\,c})$$
$$+\frac{1}{2}s_{\dot{a}}^b\,(s')_{\dot{b}}^c(i\,\bar{\kappa}_{+\,a}\,\kappa^{+\,a} - \bar{\kappa}'_{+\,a}\,\bar{\kappa}^{-\,a}) - \frac{1}{2}(s')_{\dot{a}}^a\,(s')_{\dot{b}}^b\,\bar{\kappa}_{+\,a}\,\bar{\kappa}^{-\,c}\Big) + s_{\dot{a}}^a\,\bar{s}_a^{\dot{a}}\Big(i\,\bar{\kappa}_{+\,b}\,\bar{\kappa}'^{-\,b} - i\,\kappa'_{-\,b}\,\kappa^{+\,b}$$
$$+\bar{\kappa}_{+\,b}\,\kappa^{+\,b} - \frac{1}{2}\kappa'_{-\,b}\,\bar{\kappa}'^{-\,b} - \frac{5}{4}\kappa_{-\,b}\,\bar{\kappa}^{-\,b}\Big) + s_{\dot{a}}^a\,\bar{s}_b^{\dot{a}}\Big(i\,\kappa_{-\,a}\,\kappa'^{+\,b} - i\,\bar{\kappa}_{+\,a}\,\bar{\kappa}'^{-\,b} + \frac{1}{2}\kappa'_{-\,a}\,\bar{\kappa}'^{-\,b} + \frac{1}{2}\bar{\kappa}'_{+\,a}\,\kappa'^{+\,b}$$
$$+\frac{1}{4}\kappa_{-\,a}\,\bar{\kappa}^{-\,b} + \frac{1}{4}\bar{\kappa}_{+\,a}\,\kappa^{+\,b}\Big) + (s')_{\dot{a}}^a\,\bar{s}_a^{\dot{a}}\Big(\kappa'_{-\,b}\,\bar{\kappa}^{-\,b} + \frac{i}{2}\bar{\kappa}_{+\,b}\,\bar{\kappa}^{-\,b} - \frac{1}{2}\kappa_{-\,b}\,\bar{\kappa}'^{-\,b} - \frac{1}{2}\bar{\kappa}_{+\,b}\,\kappa'^{+\,b} + \frac{1}{2}\bar{\kappa}'_{+\,b}\,\kappa^{+\,b}\Big)$$
$$-(s')_{\dot{a}}^a\,\bar{s}_b^{\dot{a}}\Big(i\,\bar{\kappa}_{+\,a}\,\bar{\kappa}^{-\,b} + \frac{1}{2}\bar{\kappa}_{+\,a}\,\kappa'^{+\,b} + \frac{1}{2}\kappa'_{-\,a}\,\bar{\kappa}^{-\,b}\Big) - \frac{1}{2}(s')_{\dot{a}}^a\,(\bar{s}')_a^{\dot{a}}\,\kappa_{-\,b}\,\bar{\kappa}^{-\,b}$$
$$+\frac{1}{2}(s')_{\dot{a}}^a\,(\bar{s}')_b^{\dot{a}}\Big(\kappa_{-\,a}\,\bar{\kappa}^{-\,b} + \bar{\kappa}_{+\,a}\,\kappa^{+\,b}\Big) + h.c.$$

The original mixing Hamiltonian is rather involved and is given by

$$-g\,\mathcal{H}^{ns}_{BF} = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(B.3)}$$

$$\frac{i}{2}y^2\,s^a_{\dot a}\,(s')^{\dot a}_a - y\,s^a_{\dot a}\,(\bar s')^{\dot a}_b\,Z^b_a - \frac{i}{4}y^2\,\bar\Psi\,\gamma^1\,\Psi' - p^2_y\big(2\,\bar\Psi\cdot\Psi + i\,\bar\Psi'\,\gamma^1\,\Psi\big) - \frac{1}{2}y'^2\big(\bar\Psi\cdot\Psi + \frac{i}{2}\bar\Psi'\,\gamma^1\,\Psi\big)$$

$$-i\frac{3}{4}p_y\,\omega_{\dot a}\,\kappa_{-a}\,\bar s^{a\,\dot a} + 3\big(\frac{i}{2}\,y\,p_{\dot a}\,\kappa_{-a}\,\bar s^{a\,\dot a} - \frac{1}{16}\,y\,\omega_{\dot a}\,\kappa_{-a}\,\bar s^{a\,\dot a}\big) + \frac{i}{4}\,y\,\omega_{\dot a}\,\bar\kappa_{+a}\,(\bar s')^{a\,\dot a} - \frac{i}{2}\,y\,\omega_{\dot a}\,\bar\kappa'_{+a}\,\bar s^{a\,\dot a}$$

$$+\frac{i}{4}\,y\,\omega'_{\dot a}\,\kappa_{-a}\,(\bar s')^{a\,\dot a} - \frac{i}{4}\,y\,\omega'_{\dot a}\,\kappa'_{-a}\,\bar s^{a\,\dot a} + \frac{i}{4}\,y'\,\omega_{\dot a}\,\kappa'_{-,a}\,\bar s^{a\,\dot a} - \frac{i}{4}\,y'\,\omega_{\dot a}\,\kappa_{-a}\,(\bar s')^{a\,\dot a} - i\frac{3}{4}\,p_y\,\omega_{\dot a}\,\bar\kappa_{+a}\,s^{a\,\dot a}$$

$$+i\frac{3}{2}\,y\,p_{\dot a}\,\bar\kappa_{+a}\,s^{a\,\dot a} + i\,\frac{3}{16}\,y\,\omega_{\dot a}\,\bar\kappa_{+a}s^{a\,\dot a} - \frac{i}{2}\,y\,\omega_{\dot a}\,\kappa'_{-a}\,s^{a\,\dot a} + \frac{i}{4}\,y\,\omega_{\dot a}\,\kappa_{-a}\,(s')^{a\,\dot a} + \frac{1}{4}\,y\,\omega'_{\dot a}\,\bar\kappa'_{+a}\,s^{a\,\dot a}$$

$$-\frac{1}{4}\,y\,\omega'_{\dot a}\,\bar\kappa_{+a}\,(s')^{a\,\dot a} - \frac{1}{4}\,y'\,\omega_{\dot a}\,\bar\kappa'_{+a}\,s^{a\,\dot a} + \frac{1}{4}\,y'\,\omega_{\dot a}\,\bar\kappa_{+a}\,(s')^{a\,\dot a} + \omega_{\dot a}\big(\bar\kappa'_{+a}\,s^{b\,\dot a} + \frac{1}{2}\bar\kappa_{+a}\,(s')^{b\,\dot a}\big)\,Z^a_b$$

$$-2\,i\,\bar\Psi_a\,\gamma^0\,\Psi^b\,\big(P_z\cdot Z - \frac{1}{2}Tr(P_z\cdot Z)\mathbb{1}\big)^a_b - \bar\Psi_a\cdot(\Psi')^b\,\big(Z\cdot Z' - \frac{1}{2}Tr(Z\cdot Z')\mathbb{1}\big)^a_b$$

$$-\frac{1}{2}\big(i\,\bar\Psi'\,\gamma^1\,\Psi + 2\,\bar\Psi\cdot\Psi\big)\,Tr(Pz\cdot Pz) - \frac{i}{8}\,\bar\Psi'\,\gamma^1\,\Psi\,Tr(Z\cdot Z) - \frac{i}{4}\,s^a_{\dot a}\,(s')^{\dot a}_a\,Tr(Z\cdot Z)$$

$$-\frac{1}{8}\big(i\,\bar\Psi'\,\gamma^1\,\Psi + 2\,\bar\Psi\cdot\Psi\big)\,Tr(Z'\cdot Z') + \frac{1}{2}(s')^a_{\dot a}\,\bar s^{\dot a}_b\big(Z\cdot Z' - \frac{1}{2}Tr(Z\cdot Z')\mathbb{1}\big)^b_a$$

$$+2i\,s^a_{\dot a}\,\bar s^{\dot a}_b\big(P_z\cdot Z - \frac{1}{2}Tr(P_z\cdot Z)\mathbb{1}\big)^b_a - \big(\kappa'_{-a}\,\bar s^{\dot a}_b + \frac{1}{2}\kappa_{-a}\,(\bar s')^{\dot a}_b\big)\omega_{\dot a}\,Z^{b\,a} + 2i\,x'^-\,\bar\Psi\,\gamma^0\,\Psi'$$

$$-4\big(i\,\bar\Psi'\,\gamma^1\,\Psi + 2\,\bar\Psi\cdot\Psi\big)\,p_{\dot a}\,\bar p^{\dot a} + 2i\,\bar\Psi\,\gamma^0\,\Psi\,p_{\dot a}\,\bar\omega^{\dot a} - \frac{1}{4}\big(i\,\bar\Psi'\,\gamma^1\,\Psi + 2\,\bar\Psi\cdot\Psi\big)\,\omega'_{\dot a}\,(\bar\omega')^{\dot a}$$

$$-i\frac{9}{16}\,\bar\Psi\,\gamma^1\,\Psi'\,\omega_{\dot a}\,\bar\omega^{\dot a} + \frac{1}{2}\big(\bar\Psi'\cdot\Psi - \bar\Psi\cdot\Psi'\big)\omega'_{\dot a}\,\bar\omega^{\dot a} + \frac{1}{4}\big(s^a_{\dot a}\,(\bar s')^{\dot b}_a - (s')^a_{\dot a}\,\bar s^{\dot b}_a\big)\,\omega_{\dot b}\,(\bar\omega')^{\dot a}$$

$$+\frac{1}{8}\big(s^a_{\dot a}\,(\bar s')^{\dot a}_a - s^a_{\dot a}\,\bar s^{\dot a}_a\big)\,\omega'_{\dot b}\,\bar\omega^{\dot b} + 2i\,s^a_{\dot a}\,s^{\dot b}_a\,\omega_{\dot b}\,\bar p^{\dot a} - i\,s^a_{\dot a}\,\bar s^{\dot a}_a\,\omega_{\dot b}\,\bar p^{\dot b} + \frac{i}{4}\,s^a_{\dot a}\,(s')^{\dot a}_a\,\omega_{\dot b}\,\bar\omega^{\dot b} + h.c.$$

Note the slight asymmetry between the $\kappa^\pm$ fields. This is due to the fact that we have not considered the kinetic terms of the fermions, with witch one should augment the non shifted Hamiltonian.

C. Fermionic shift

The fermionic shift has to be implemented on the quadratic and cubic Hamiltonian in (3.4) and (B.1). In order to attain this one need the explicit form of the fermionic shift. Starting from (3.9), one can write

$$\frac{1}{g}\mathcal{L}^\eta_{Kin} = \frac{1}{2}Str\,\dot\eta\Big\{[\eta, G_t\,\boldsymbol{\pi}\,G^{-1}_t] + \frac{1}{4}\big([G_t\,\boldsymbol{\pi}\,G^{-1}_t, \eta^3] + \eta\,[G_t\,\boldsymbol{\pi}\,G^{-1}_t, \eta]\,\eta\big)\quad\text{(C.1)}$$

$$-i\,\kappa\,G_t\,\Upsilon\,G^{-1}_t\Big(\frac{i}{2}[\Sigma_-, \eta]x'^- + \eta' - \frac{1}{2}\eta\,\eta'\,\eta\Big)G_t\,\Upsilon^{-1}\,G^{-1}_t$$

$$+\frac{i}{2}\,\kappa\,\eta\,G_t\,\Upsilon\,G^{-1}_t\Big(\frac{i}{2}[\Sigma_-, \eta]x'^- + \eta' - \frac{1}{2}\eta\,\eta'\,\eta\Big)G_t\,\Upsilon^{-1}\,G^{-1}_t\,\eta\Big\} + \mathcal{O}(\eta^6).$$

Where the leading order term is the quadratic kinetic term and the higher order terms are just the function $\widetilde{\Phi}(\eta)$ introduced earlier, which together with its self interaction terms constitute the fermionic shift.

Since we do a perturbative analysis up to quartic order, the presence of quadratic fermionic terms in the above expressions imply that we need $\pi_\pm$ to quadratic order only[28], from (2.30) we find

$$g\,\pi_- = \frac{i}{4}\Big(p_y^2 + \frac{1}{2}Tr(P_z \cdot P_z) + 4\,p_{\dot{a}}\,\bar{p}^{\dot{a}} + \omega'_{\dot{a}}\,(\bar{\omega}')^{\dot{a}} + y'^2 + \frac{1}{2}Tr(Z' \cdot Z')\Big), \qquad \text{(C.2)}$$

$$g\,\pi_+ = \frac{i}{4} + \frac{i}{16}\Big(y^2 - \frac{1}{2}Tr(Z \cdot Z) + \frac{1}{4}\omega_{\dot{a}}\,\bar{\omega}^{\dot{a}}\Big) + \frac{1}{4}\Big(\bar{\Psi}'\gamma^1\Psi - \bar{\Psi}\gamma^1\Psi' - \frac{i}{2}\bar{\Psi}\cdot\Psi\Big).$$

Combining the solutions for $\pi_\pm$ and the transverse components of π in (2.28) one can solve for the fermionic shift (3.14) explicitly. As should be clear, the explicit form in components is quite complicated. Nevertheless, it is a straightforward task to obtain the shift for each coordinates by inverting the expressions (2.24).

To obtain the full shift that also removes the $Str\,\Phi_2\,\widetilde{\Phi}_2$ term, one need to isolate the $\dot{\eta}$ part and add this contribution to (3.14). The terms from $Str\,\Phi_2\,\widetilde{\Phi}_2$ without a $\dot{\eta}$ dependence will introduce corrections to π_t which one also need to determine explicitly. Having done all this, one can implement the full shift in the original Hamiltonian, together with the corrections to π, and determine the full mixing part of the shifted Hamiltonian. Needless to say, all this will be a rather involved procedure and is beyond the scope of this paper.

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[28]This is only true for the fermionic kinetic term. In the full Lagrangian π_- is needed to quartic order.

References

[1] J. H. Schwarz, JHEP **0411**, 078 (2004) [arXiv:hep-th/0411077]. J. Bagger and N. Lambert, Phys. Rev. D **75**, 045020 (2007) [arXiv:hep-th/0611108]. J. Bagger and N. Lambert, Phys. Rev. D **77**, 065008 (2008) [arXiv:0711.0955 [hep-th]]. A. Gustavsson, arXiv:0709.1260 [hep-th]. J. Gomis, D. Rodriguez-Gomez, M. Van Raamsdonk and H. Verlinde, JHEP **0808**, 094 (2008) [arXiv:0806.0738 [hep-th]]. J. Bagger and N. Lambert, JHEP **0802**, 105 (2008) [arXiv:0712.3738 [hep-th]]. D. Gaiotto and X. Yin, JHEP **0708**, 029 (2007) [arXiv:0707.3437 [hep-th]]. D. Gaiotto and E. Witten, arXiv:0804.2907 [hep-th]. M. A. Bandres, A. E. Lipstein and J. H. Schwarz, JHEP **0805**, 025 (2008) [arXiv:0803.3242 [hep-th]]. M. Van Raamsdonk, JHEP **0805**, 105 (2008) [arXiv:0803.3803 [hep-th]]. N. Lambert and D. Tong, Phys. Rev. Lett. **101**, 041602 (2008) [arXiv:0804.1114 [hep-th]]. S. Benvenuti, D. Rodriguez-Gomez, E. Tonni and H. Verlinde, arXiv:0805.1087 [hep-th]. M. A. Bandres, A. E. Lipstein and J. H. Schwarz, JHEP **0807**, 117 (2008) [arXiv:0806.0054 [hep-th]]. J. Bagger and N. Lambert, arXiv:0807.0163 [hep-th].

[2] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, JHEP **0810**, 091 (2008) [arXiv:0806.1218 [hep-th]].

[3] J. A. Minahan and K. Zarembo, JHEP **0809**, 040 (2008) [arXiv:0806.3951 [hep-th]].

[4] T. Nishioka and T. Takayanagi, JHEP **0808**, 001 (2008) [arXiv:0806.3391 [hep-th]].

[5] C. G. . Callan, H. K. Lee, T. McLoughlin, J. H. Schwarz, I. Swanson and X. Wu, Nucl. Phys. B **673**, 3 (2003) [arXiv:hep-th/0307032].

[6] C. G. . Callan, T. McLoughlin and I. Swanson, Nucl. Phys. B **694**, 115 (2004) [arXiv:hep-th/0404007].

[7] C. G. . Callan, T. McLoughlin and I. Swanson, Nucl. Phys. B **700**, 271 (2004) [arXiv:hep-th/0405153].

[8] D. Gaiotto, S. Giombi and X. Yin, arXiv:0806.4589 [hep-th].

[9] G. Arutyunov and S. Frolov, JHEP **0809**, 129 (2008) [arXiv:0806.4940 [hep-th]].

[10] B. . j. Stefanski, Nucl. Phys. B **808**, 80 (2009) [arXiv:0806.4948 [hep-th]].

[11] K. Zarembo, arXiv:0903.1747 [hep-th].

[12] P. Fre and P. A. Grassi, arXiv:0807.0044 [hep-th].

[13] N. Gromov and P. Vieira, arXiv:0807.0777 [hep-th].

[14] N. Gromov and V. Mikhaylov, arXiv:0807.4897 [hep-th].

[15] N. Gromov and P. Vieira, arXiv:0807.0437 [hep-th].

[16] L. F. Alday, G. Arutyunov and D. Bykov, arXiv:0807.4400 [hep-th].

[17] T. McLoughlin and R. Roiban, arXiv:0807.3965 [hep-th].

[18] C. Krishnan, JHEP **0809**, 092 (2008) [arXiv:0807.4561 [hep-th]].

[19] T. McLoughlin, R. Roiban and A. A. Tseytlin, arXiv:0809.4038 [hep-th].

[20] C. Kristjansen, M. Orselli and K. Zoubos, arXiv:0811.2150 [hep-th].

[21] C. Kalousios, C. Vergu and A. Volovich, arXiv:0905.4702 [hep-th].

[22] D. Bak, H. Min and S. J. Rey, arXiv:0904.4677 [hep-th].

[23] D. Bak and S. J. Rey, JHEP **0810**, 053 (2008) [arXiv:0807.2063 [hep-th]].

[24] D. Bak, D. Gang and S. J. Rey, JHEP **0810**, 038 (2008) [arXiv:0808.0170 [hep-th]].

[25] B. I. Zwiebel, arXiv:0901.0411 [hep-th].

[26] P. Sundin, JHEP **0902**, 046 (2009) [arXiv:0811.2775 [hep-th]].

[27] D. Bykov, arXiv:0904.0208 [hep-th].

[28] R. Suzuki, JHEP **0905**, 079 (2009) [arXiv:0902.3368 [hep-th]].

[29] C. Ahn and R. I. Nepomechie, JHEP **0903**, 144 (2009) [arXiv:0901.3334 [hep-th]].

[30] J. A. Minahan, W. Schulgin and K. Zarembo, JHEP **0903**, 057 (2009) [arXiv:0901.1142 [hep-th]].

[31] F. Spill, arXiv:0810.3897 [hep-th].

[32] A. Hamilton, J. Murugan, A. Prinsloo and M. Strydom, arXiv:0901.0009 [hep-th].

[33] H. Dimov and R. C. Rashkov, arXiv:0908.2218 [hep-th].

[34] M. Schimpf and R. C. Rashkov, arXiv:0908.2246 [hep-th].

[35] M. Dukalski and S. J. van Tongeren, Phys. Rev. D **80**, 046005 (2009) [arXiv:0906.0706 [hep-th]].

[36] G. Arutyunov and S. Frolov, J. Phys. A **42**, 254003 (2009) [arXiv:0901.4937 [hep-th]].

[37] N. Berkovits, M. Bershadsky, T. Hauer, S. Zhukov and B. Zwiebach, Nucl. Phys. B **567**, 61 (2000) [arXiv:hep-th/9907200].

[38] N. Beisert, B. Eden and M. Staudacher, J. Stat. Mech. **0701**, P021 (2007) [arXiv:hep-th/0610251].

[39] N. Beisert, Adv. Theor. Math. Phys. **12**, 945 (2008) [arXiv:hep-th/0511082].

[40] N. Beisert and M. Staudacher, Nucl. Phys. B **727**, 1 (2005) [arXiv:hep-th/0504190].

[41] N. Dorey, D. M. Hofman and J. M. Maldacena, Phys. Rev. D **76**, 025011 (2007) [arXiv:hep-th/0703104].

[42] C. Ahn and R. I. Nepomechie, JHEP **0809**, 010 (2008) [arXiv:0807.1924 [hep-th]].

[43] G. Grignani, T. Harmark and M. Orselli, arXiv:0806.4959 [hep-th].

[44] D. Berenstein and D. Trancanelli, arXiv:0808.2503 [hep-th].

[45] D. Astolfi, V. G. M. Puletti, G. Grignani, T. Harmark and M. Orselli, arXiv:0807.1527 [hep-th].

[46] G. Arutyunov and S. Frolov, JHEP **0601**, 055 (2006) [arXiv:hep-th/0510208].

[47] L. F. Alday, G. Arutyunov and S. Frolov, JHEP **0601**, 078 (2006) [arXiv:hep-th/0508140].

[48] S. Frolov, J. Plefka and M. Zamaklar, J. Phys. A **39**, 13037 (2006) [arXiv:hep-th/0603008].

[49] J. Gomis, D. Sorokin and L. Wulff, arXiv:0811.1566 [hep-th].

[50] P. A. Grassi, D. Sorokin and L. Wulff, JHEP **0908**, 060 (2009) [arXiv:0903.5407 [hep-th]].

[51] G. Arutyunov, S. Frolov, J. Plefka and M. Zamaklar, J. Phys. A **40**, 3583 (2007) [arXiv:hep-th/0609157].

[52] A. Hentschel, J. Plefka and P. Sundin, JHEP **0705**, 021 (2007) [arXiv:hep-th/0703187].

[53] G. Arutyunov, S. Frolov and M. Zamaklar, Nucl. Phys. B **778**, 1 (2007) [arXiv:hep-th/0606126].

[54] J. M. Maldacena and I. Swanson, Phys. Rev. D **76**, 026002 (2007) [arXiv:hep-th/0612079].

[55] M. Kreuzer, R. C. Rashkov and M. Schimpf, Eur. Phys. J. C **60**, 471 (2009) [arXiv:0810.2008 [hep-th]].

[56] D. E. Berenstein, J. M. Maldacena and H. S. Nastase, AIP Conf. Proc. **646**, 3 (2003).

[57] Grassmannops.m, Jeremy Michelson, http://www.physics.ohio-state.edu/ jeremy/mathematica/grassmannOps/